Le hasard fait bien les choses

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Inria

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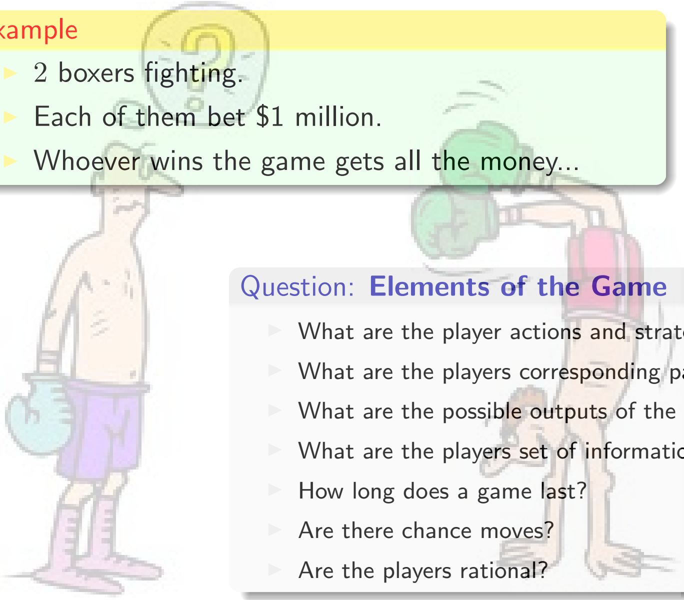
What is Game Theory and what is it for?

Definition (Roger Myerson, *"Game Theory, Analysis of Conflicts"*)

"Game theory can be defined as the study of mathematical models of conflict and cooperation between intelligent rational decision-makers. Game theory provides general mathematical techniques for analyzing situations in which two or more individuals make decisions that will influence one another's welfare."

- Branch of optimization
- Multiple actors with different objectives
- Actors interact with each others

Example of Game



Example

- 2 boxers fighting.
- Each of them bet \$1 million.
- Whoever wins the game gets all the money...

Question: **Elements of the Game**

- What are the player actions and strategies?
- What are the players corresponding payoffs?
- What are the possible outputs of the game?
- What are the players set of information?
- How long does a game last?
- Are there chance moves?
- Are the players rational?

Game Theory and Nobel Prices in Economy

- Alvin Roth (2012, 1951) – experimental GT
- Lloyd Shapley (2012, 1923) – fair sharing, potential games
- Roger B. Myerson (2007, 1951) – eq. in dynamic games
- Leonid Hurwicz (2007, 1917-2008) – incentives
- Eric S. Maskin (2007, 1950) – mechanism design
- Robert J. Aumann (2005, 1930) – correlated equilibria
- Thomas C. Schelling (2005, 1921) – bargaining
- William Vickrey (1996, 1914-1996) – pricing
- Robert E. Lucas Jr. (1995, 1937) – rational expectations
- John C. Harsanyi (1994, 1920-2000) – Bayesian games, eq. selection
- John F. Nash Jr. (1994, 1928) – NE, NBS
- Reinhard Selten (1994, 1930) – Subgame perf. eq., bounded rationality
- Kenneth J. Arrow (1972, 1921) – Impossibility theorem
- Paul A. Samuelson (1970, 1915-2009) – thermodynamics to econ.

(Jorgen Weibull - Chairman 2004-2007)

(more info on http://lcm.csa.iisc.ernet.in/gametheory/nobel.html)

Example of successful applications

Economy:

- Pricing mechanisms
- Auctions

Politics:

- Fight against terrorism
- Negotiation and dispute resolution, bargaining
- Effect of electoral rules to politicians' strategies

Biology:

- Cancer cells propagation
- Genetics and population evolution

And many others:

- Evolutionary psychology (social sciences)
- Intellectual right properties (law)
- Policy responses to global warming and climate change...

La théorie des jeux et les systèmes (informatiques) distribués

- Rien à voir avec les jeux vidéos

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Popularité croissante dans les grands systèmes distribués depuis les années 90 du fait de:

- L'augmentation du nombre des protagonistes
- L'accroissement et la complexification des systèmes
- La dynamicité

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$\Rightarrow$ on a besoin de méthodes automatisées pour concevoir, gérer les systèmes et évaluer les performances

Outline

1. Individual Versus Collective Interest
    - Matrix Games - Nash Equilibria
    - Population Games - Wardrop Equilibria
    - Conclusion
    - Application: Performance Analysis

2. Designing Efficient Control Mechanisms
    - Objective: Fair Sharing of Resources
    - Direct Method: Protocol Implementation
    - Indirect Method: Modifying the game

3. Conclusion

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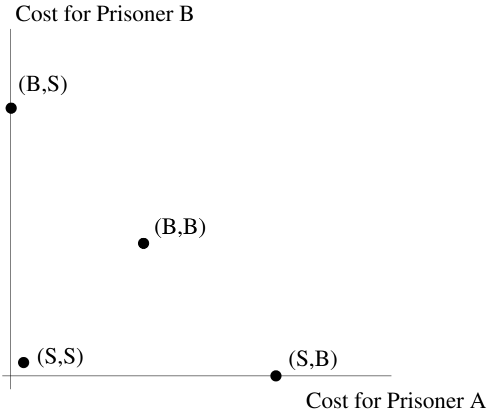
The Prisoner Dilemma

	Prisoner B stays Silent	Prisoner B Betrays
A stays Silent	Each serves 6 months	Prisoner A: 10 years Prisoner B: goes free
A Betrays	Prisoner A goes free Prisoner B: 10 years	Each serves 5 years

What is the best interest of each prisoner?

What is the output (Nash Equilibrium) of the game?

The Prisoner Dilemma - Cost Space



What are the optimal points?

What is the equilibrium?

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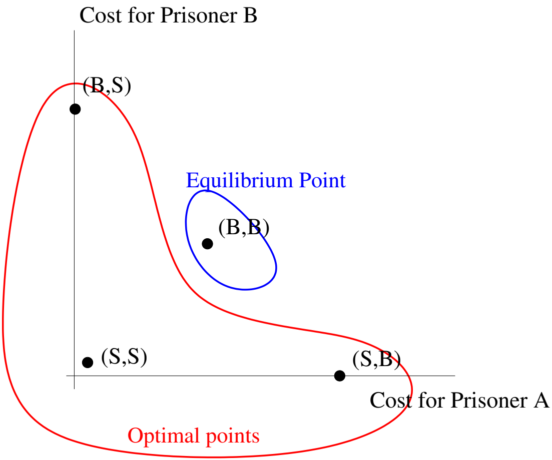

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Game in Normal Form

Definition: (Finite or Matrix) Game.

- N players, finite number of actions
- Payoffs of players (depend of each other actions and) are real valued
- Stable points are called Nash Equilibria

Definition: Nash Equilibrium.

In a NE, no player has incentive to unilaterally modify his strategy.

strategy

payoff

s^* is a Nash equilibrium iff:

$$\forall p, \forall s_p, u_p(s_1^*, \dots, s_p^*, \dots s_n^*) \geq u_p(s_1^*, \dots, s_p, \dots, s_n^*)$$

In a compact form:

$$\forall p, \forall s_p, u_p(s_{-p}^*, s_p^*) \geq u_p(s_{-p}^*, s_p)$$

Nash Equilibrium: Examples

Find the Nash equilibria of these games (with pure strategies)

The prisoner dilemma

	collaborate	deny
collaborate	$(1,1)$	$(3,0)$
deny	$(0,3)$	$(2,2)$

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Battle of the sexes

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Rock-Scisor-Paper

$1/2$	P	R	S
P	$(0,0)$	$(1,-1)$	$(-1,1)$
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⇒ No equilibrium

Mixed Nash Equilibria

Definition: **Mixed Strategy Nash Equilibria**.

A mixed strategy for player i is a probability distribution over the set of pure strategies of player i.
An equilibrium in mixed strategies is a strategy profile σ^* of mixed strategies such that: $\forall p, \forall \sigma_i, u_p(\sigma^*_{-p}, \sigma^*_p) \geq u_p(\sigma^*_{-p}, \sigma_p)$.

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Consequence:

- The players mixed strategies are independant randomizations.
- In a finite game, $u_p(\sigma) = \sum_a (\prod_{p'} \sigma_{p'}(a_{p'})) u_i(a)$.
- In a finite game, σ^* is a Nash equilibrium iff $\forall a_i$ in the support of σ^*_i, a_i is a best response to σ^*_{-i}.

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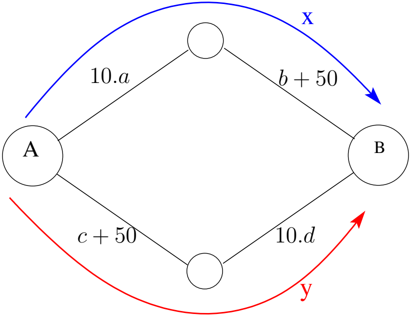
The Braess Paradox

Question: A flow of users goes from A to B, with rate of 6 (thousands of people / sec). Each driver has two possible routes to go from A to B. Who takes which route?



- 2 possibles routes
- the needed time is a function of the number of cars on the road (congestion)

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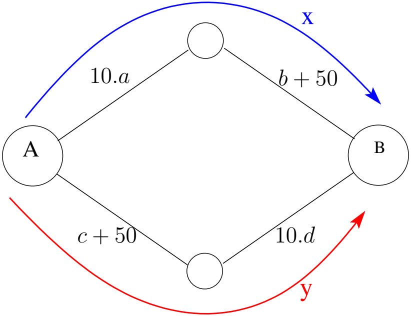

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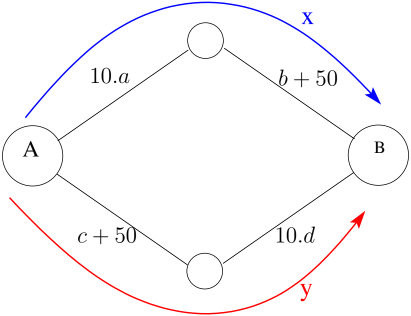

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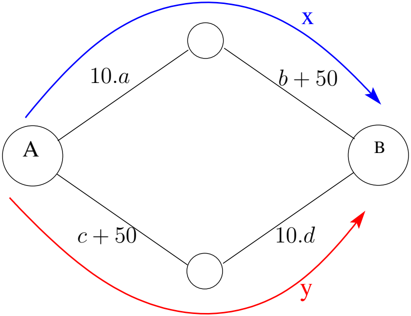

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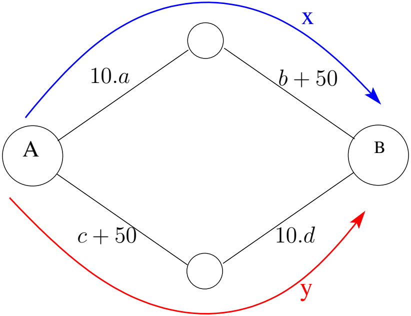

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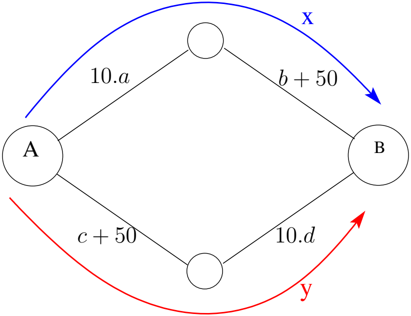

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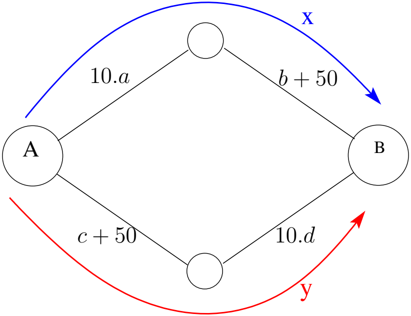

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Constraint: $x + y = 6$

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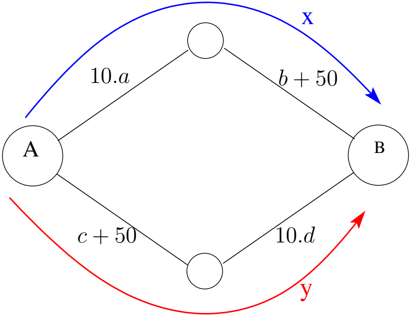

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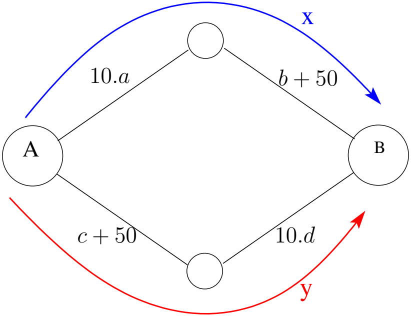

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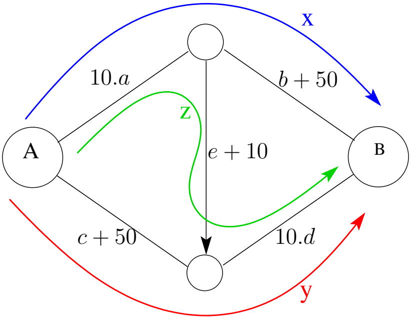
We get $x = y = 3$ and everyone receives 83

The Braess Paradox

A new road is opened! What happens?



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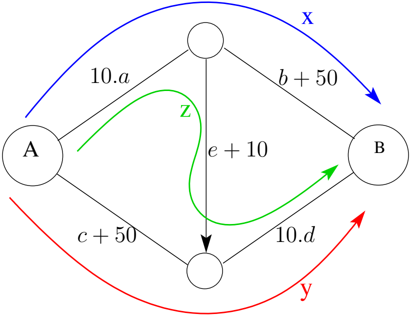

If noone takes it, it cost is

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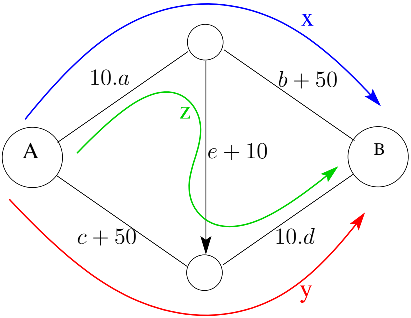

If noone takes it, it cost is 70! so rational users will take it... ☺

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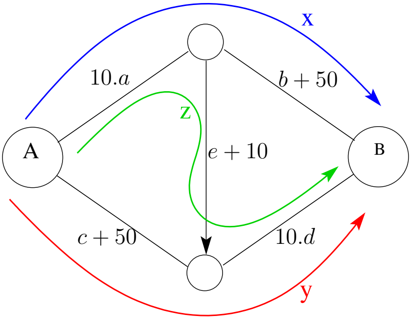

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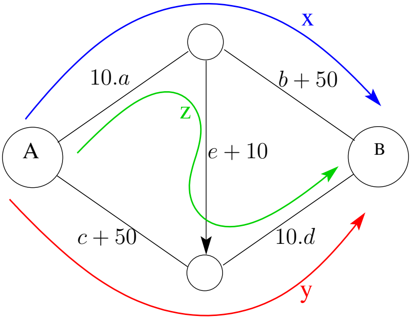

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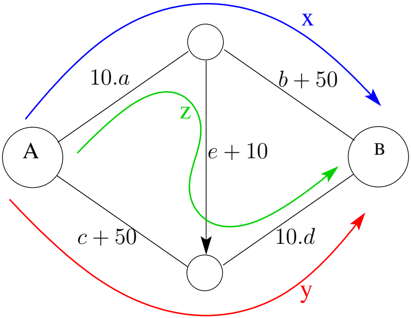

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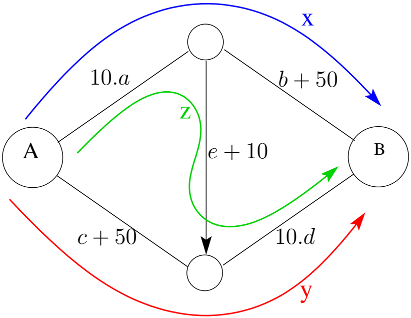

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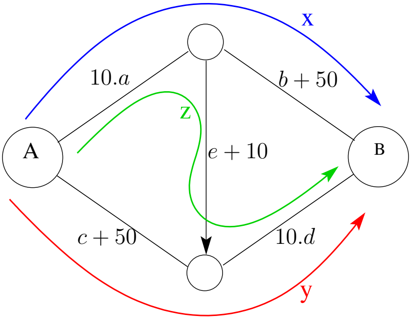

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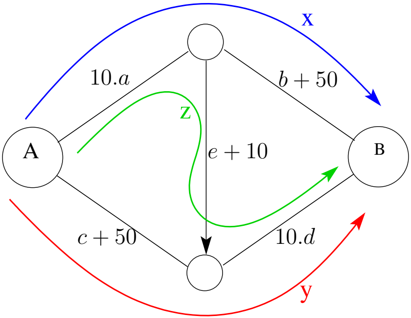

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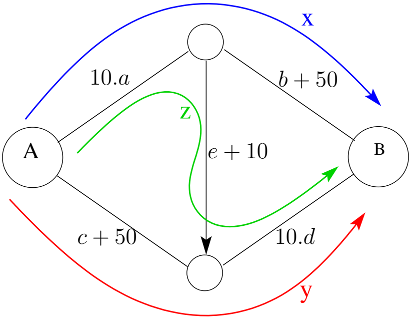

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Cost of "new" route:

$10 * x + 10 * y + 21 * z + 10$

Conclusion?

We get $x = y = z = 2$ and everyone gets a cost of 92!

The Braess Paradox

In le New York Times, 25 Dec., 1990, Page 38, What if They Closed 42d Street and Nobody Noticed?, By GINA KOLATA:

ON Earth Day this year, New York City's Transportation Commissioner decided to close 42d Street, which as every New Yorker knows is always congested. "Many predicted it would be doomsday," said the Commissioner, Lucius J. Riccio. "You didn't need to be a rocket scientist or have a sophisticated computer queuing model to see that this could have been a major problem." But to everyone's surprise, Earth Day generated no historic traffic jam. Traffic flow actually improved when 42d Street was closed.

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Efficiency versus (Individual) Stability

Prisoner Dilemma / Braess paradox show:

- Inherent conflict between individual interest and global interest
- Inherent conflict between stability and optimality

Typical problem in economy: free-market economy versus regulated economy.

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- Inherent conflict between individual interest and global interest
- Inherent conflict between stability and optimality

Typical problem in economy: free-market economy versus regulated economy.

↯ Suppose that you are a network operator. The different users compete to access the different system resources
Should you intervene?

- NO if the Nash Equilibria exhibit good performance
- YES otherwise

Efficiency versus (Individual) Stability

"Free-Market":

Efficiency versus (Individual) Stability

"Regulated Market":

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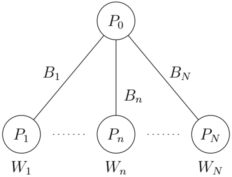
Application: Assessing the efficiency of equilibria

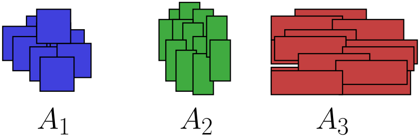
Example: Multiple Bag-of-Task Applications in Distributed Platforms

- Multiple applications execute concurrently on heterogeneous platforms and compete for CPU and network resources.
- A fair sharing of resources amongst users is done at the system layer (network, OS).
- We analyze the behavior of non-cooperative schedulers that maximize their own utility.

Master-worker platform:



Applications' profiles:



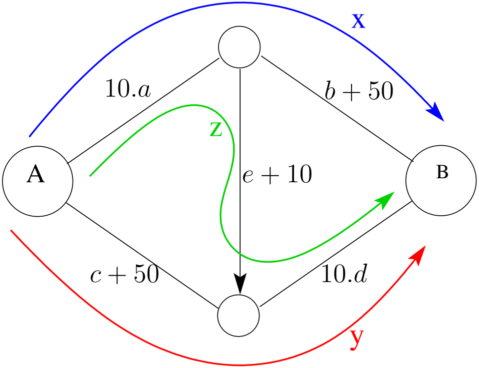
Such applications are typical desktop grid applications (SETI@home, Einstein@Home, processing data of the Large Hadron Collider)...

Application: Assessing the efficiency of equilibria

Example: Packet Routing in Networks

A routing problem is a triplet:

- A graph $G = (N, A)$ (the network)
- A set of flows d_k, $k \in K$ and $K \subset N \times N$ (user demands)
- latency funtions ℓ_a for each link



Theorem 2.

In networks with affine costs [Roughgarden & Tardos, 2002],

$$C^{WE} \leq \frac{4}{3} C^{SO}.$$

⇒ In affine routing, selfishness leads to a near optimal point.

Suppose that the system could be in 2 states w_1 and w_2, with probability $P(w_1) = P(w_2) = 1/2$.

(w_1)	a	b
a	$(0, 0)$	$(6, -3)$
b	$(-3, 6)$	$(5, 5)$

(w_2)	a	b
a	$(-20, -20)$	$(-7, -16)$
b	$(-16, -7)$	$(-5, -5)$

What is the Nash Equilibria if:

- No player knows the system's state:
- Both players are informed:
- Only player 1 knows:

Assessing the efficiency of equilibria

Example: Measuring the influence of information

Suppose that the system could be in 2 states w_1 and w_2, with probability $P(w_1) = P(w_2) = 1/2$.

(w_1)	a	b
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(w_2)	a	b
a	$(-20, -20)$	$(-7, -16)$
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What is the Nash Equilibria if:

- No player knows the system's state:
  EN: (b, b), utility :$(0, 0)$
- Both players are informed:
  EN: $((a, a)|w1), ((b, b)|w2)$, utilité: $(-2.5, -2.5)$
- Only player 1 knows:
  EN: $((a, a)|w1)$, $((b, a)|w2)$, utilité $(-8, -3, 5)$

Information can be detrimental!

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Bargaining Theory

- Aims at predicting the outcome of a bargain between 2 (or more) players
- The players are bargaining over a set of goods
- To each good is associated for each player a utility (for instance real valued)

Assumptions:

- Players have identical bargaining power
- Players have identical bargaining skills

Then, players will eventually agree on an point considered as “fair” for both of them.

The Nash Solution

Let S be a feasible set, closed, convex, (u^*, v^*) a point in this set, enforced if no agreement is reached.
A fair solution is a point $\phi(S, u^*, v^*)$ satisfying the set of axioms:

1. (Individual Rationality) $\phi(S, u^*, v^*) \geq (u^*, v^*)$ (componentwise)
2. (Feasibility) $\phi(S, u^*, v^*) \in S$
3. (Pareto-Optimality) $\forall (u, v) \in S, (u, v) \geq \phi(S, u^*, v^*) \rightarrow (u, v) = \phi(S, u^*, v^*)$
4. (Independence of Irrelevant Alternatives) $\phi(S, u^*, v^*) \in T \subset S \Rightarrow \phi(S, u^*, v^*) = \phi(T, u^*, v^*)$
5. (Independence of Linear Transformations) Let $F(u, v) = (\alpha_1 u + \beta_1, \alpha_2 v + \beta_2)$, $T = F(S)$, then $\phi(T, F(u^*, v^*)) = F(\phi(S, u^*, v^*))$
6. (Symmetry) If S is such that $(u, v) \in S \Leftrightarrow (v, u) \in S$ and $u^* = v^*$ then $\phi(S, u^*, v^*) \stackrel{\text{def}}{=} (a, b)$ is such that $a = b$

The Nash Solution

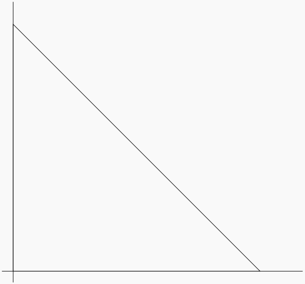
Proposition: Nash Bargaining Solution

There is a unique solution function ϕ satisfying all axioms:

$$\phi(S, u^*, v^*) = \max_{u,v}(u - u^*)(v - v^*)$$

Proof.

First case: Positive quadrant right isosceles triangle



Second Case: General case

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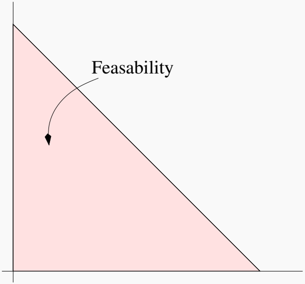

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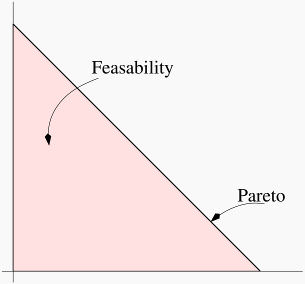

Second Case: General case

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The Nash Solution

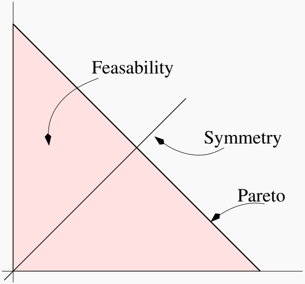
Proposition: **Nash Bargaining Solution**

There is a unique solution function ϕ satisfying all axioms:

$$\phi(S, u^*, v^*) = \max_{u,v}(u - u^*)(v - v^*)$$

Proof.

First case: Positive quadrant right isosceles triangle



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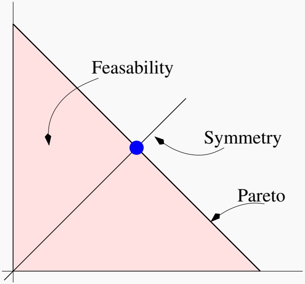

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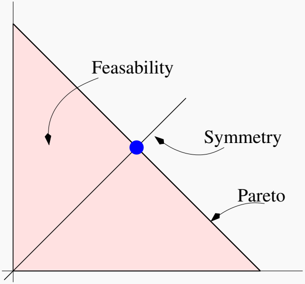

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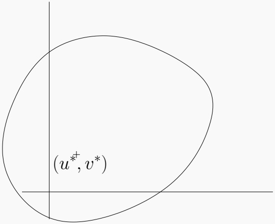

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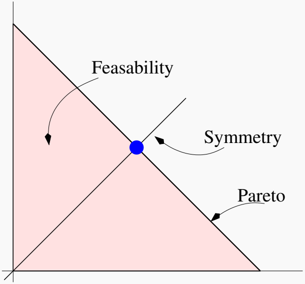

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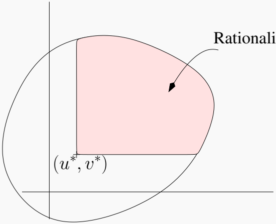

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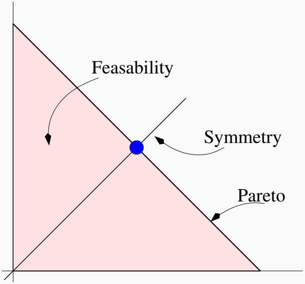

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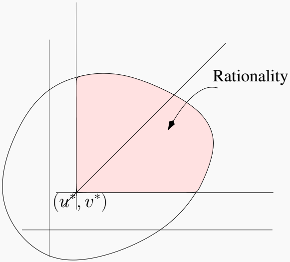

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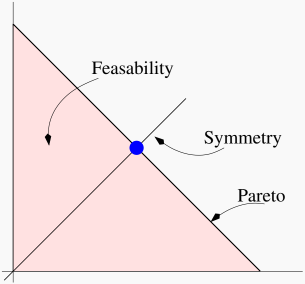

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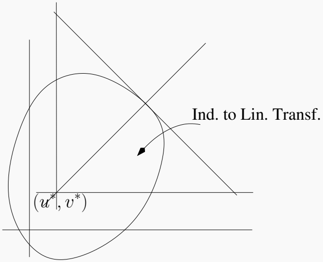

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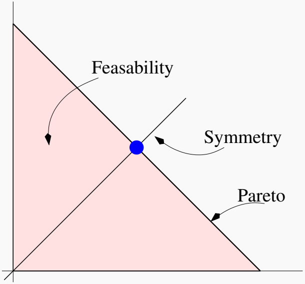

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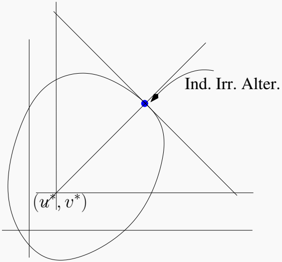

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Axiomatic Definition VS Optimization Problem

1. Individual Rationality
2. Feasibility
3. Pareto-Optimality
5. Independence of Linear Transformations
6. Symmetry

Axiomatic Definition VS Optimization Problem

1. Individual Rationality
2. Feasibility
3. Pareto-Optimality
5. Independence of Linear Transformations
6. Symmetry

$+$

4. Independant to irrelevant alternatives Nash (NBS) / Proportional Fairness $\prod (u_i - u_i^d)$
4. Monotony Raiffa-Kalai-Smorodinsky / max-min Recursively $\max\{u_i | \forall j, u_i \leq u_j\}$
4. Inverse Monotony Thomson / global Optimum (Social welfare) $\max \sum u_i$

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The Flow Control Problem

Imagine a system with:

- n individual users aiming at optimizing their throughput x_n
- A routing matrix A giving the set of paths followed by each connection: $A_{i,j} = \begin{cases} 1 & \text{if connection } i \text{uses link } j \\ 0 & \text{otherwise} \end{cases}$
- Capacity constraints on each link C_ℓ

- What is the Nash equilibrium of the game? What protocol does it corresponds to?
- How can we implement fairness in a distributed way?

The Flow Control Problem:

The Non Cooperative Game

Example: A simple network with 3 links

(namSimple.mpeg)

$n_{0\to 2} = 2$, $n_{1\to 2} = 3$, $n_{2\to 3} = 4$

Throughput of flow i:

$$\frac{\lambda_i.\text{capa}}{\lambda_1 + \lambda_2}$$

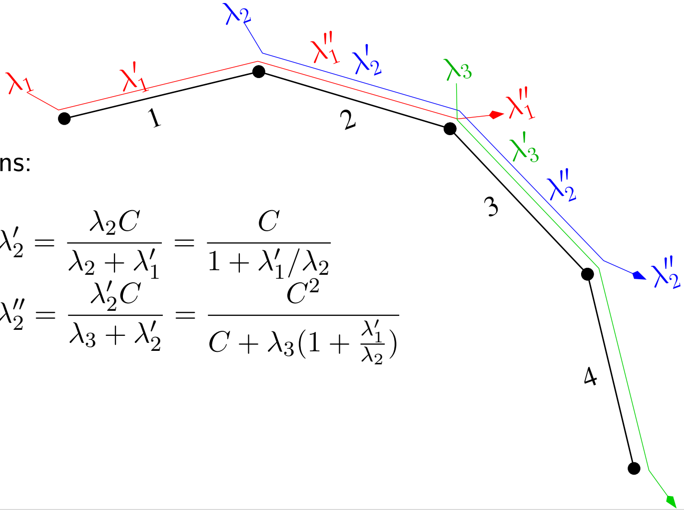
The Flow Control Problem:

The Non Cooperative Game

Hypothesis:

Ring topology network, N identical links with capacity C.

Source i uses links i and $i+1$ (mod N)



Link equations:

$$\begin{cases} \lambda_2' = \dfrac{\lambda_2 C}{\lambda_2 + \lambda_1'} = \dfrac{C}{1 + \lambda_1'/\lambda_2} \\ \lambda_2'' = \dfrac{\lambda_2' C}{\lambda_3 + \lambda_2'} = \dfrac{C^2}{C + \lambda_3(1 + \frac{\lambda_1'}{\lambda_2})} \end{cases}$$

The Flow Control Problem:

The Non Cooperative Game

Hypothesis $\lambda >> C$

Exit throughput of flow i:

(namUDP.mpeg)

$$\lambda'' = \frac{C^2}{C + \lambda(1 + \lambda'/\lambda)}, \qquad \text{and}$$

$$\frac{\lambda'}{\lambda} = \frac{1}{2}\left(\sqrt{1 + \frac{4C}{\lambda}} - 1\right) \sim \frac{C}{\lambda}$$

Then $\lambda'' \sim \dfrac{C^2}{\lambda}$

The Flow Control Problem:

The Non Cooperative Game

This is network collapse:

- The network is full
- Little or no useful information is going through (here $\lambda'' \sim \frac{C^2}{\lambda} \rightarrow_{\lambda\rightarrow\infty} 0$)

Observed in 1984 (cf RFC 896) with TCP flows: the protocol detects a loss, so it retransmits the packet, hence increasing its incoming throughput...

Since then a flow control mechanism has been imprented in TCP ☺

Why hadn't we observe this kind of phenomena before with telephony?

The Optimal Flow Problem

- N individual users aiming at optimizing their throughput x_n
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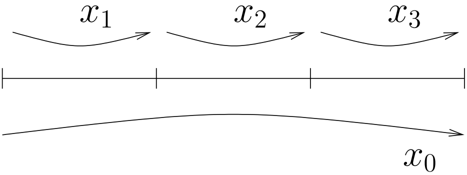
- Capacity constraints on each link C_ℓ
- User utility function $f_n : \mathbb{R}^+ \to \mathbb{R}^+$ that are increasing and strictly concave.

The flow control problem is:

$$\max_x \underbrace{\boxed{\sum_n f_n(x_n)}}_{\boxed{\text{fairness aggregation function}}} \qquad \text{s.t.} \quad \underbrace{\boxed{\forall \ell, (Ax)_\ell - C_\ell \leq 0 \text{ and } x \geq 0}}_{\boxed{\text{system constraints}}}$$

Example: The Flow Control Problem

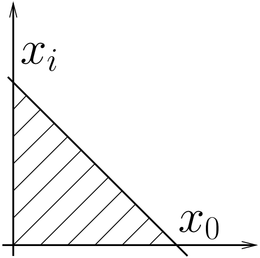
4 connections / 3 links.



$$\begin{cases} x_1 + x_0 \leq 1, \\ x_2 + x_0 \leq 1, \\ x_3 + x_0 \leq 1. \end{cases}$$

$\Rightarrow$ 4 variables and 3 (in)equalities.

How to choose x_0 among the Pareto optimal points?



(Nota: in this case the utility set is the same as the strategy set)

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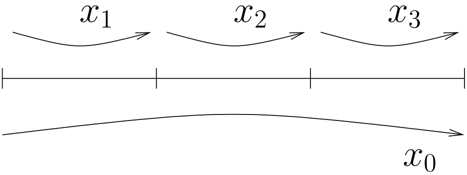

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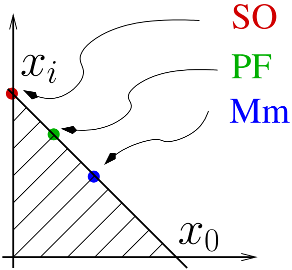

How to choose x_0 among the Pareto optimal points?

$$\begin{cases} x_0 = 0.5, \\ x_1 = x_2 = x_3 = 0.5 \end{cases}$$ Max-Min fairness

$$\begin{cases} x_0 = 0, \\ x_1 = x_2 = x_3 = 1 \end{cases}$$ Social Optimum

$$\begin{cases} x_0 = 0.25, \\ x_1 = x_2 = x_3 = 0.75 \end{cases}$$ Proportionnal Fairness

(Nota: in this case the utility set is the same as the strategy set)

The Optimal Flow Problem

- N individual users aiming at optimizing their throughput x_n
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How to (efficiently and in a distributed manner) solve this?

A possible implementation: TCP Reno / RED

Flow control algorithm: AIMD (Additive Increase Multiplicative Decrease)

- Window w_n: number of packets of a connection that can be outstanding at any time (i.e. for which no ack has been received yet)
- The Round Trip Time (RTT_n) of connection n (supposed independant of the load)
- Additive increase: at each RTT, increase the window size by 1 if there is no mark
- Multiplicative decrease: at each marked packet, halve the window size

A possible implementation: TCP Reno / RED

Flow control algorithm: AIMD (Additive Increase Multiplicative Decrease)

- Source rate: $x_n(t) = w_n(t)/RTT_n$
- Loss probability: $q_n = 1 - \prod_{\ell, A_{n,\ell}=1} (1 - p_\ell) \approx \sum_{\ell, A_{n,\ell}=1} p_\ell$
- Between two packet emissions:

$$\Delta t \approx \frac{1}{x_n(t)} = \frac{RTT_n}{w_n(t)}$$

$$w_n(t + \Delta t) - w_n(t) \approx \frac{1}{w_n(t)} \underbrace{(1 - q_n(t))}_{\text{received packet}} - \frac{w_n(t)}{2} \underbrace{q_n(t)}_{\text{lost packet}}$$

$$\Rightarrow \frac{dx_n}{dt}(t) = \frac{1 - q_n(t)}{\mathrm{RTT}_n^2} - \frac{1}{2} q_n(t) x_n^2(t)$$

TCP Reno: Conclusion

Hence, the AIMD flow control of TCP follows dynamics:

$$\frac{dx_n}{dt}(t) = \frac{1 - q_n(t)}{\mathsf{RTT}_n^2} - \frac{1}{2} q_n(t) x_n^2(t)$$

Using tools from control theory (Lyapunov function), one can establish that it maximizes over x function

$$W(x) = \sum_n \underbrace{\frac{\sqrt{2}}{\mathsf{RTT}_n} \mathsf{atan}\left(\frac{x_n \mathsf{RTT}_n}{\sqrt{2}}\right)}_{\text{Utility } f_n(x_n)} - \sum_\ell \underbrace{\int_0^{\sum_m A_{m,\ell} x_m} p_\ell(y) dy}_{\text{link cost } \mathsf{LC}_\ell}$$

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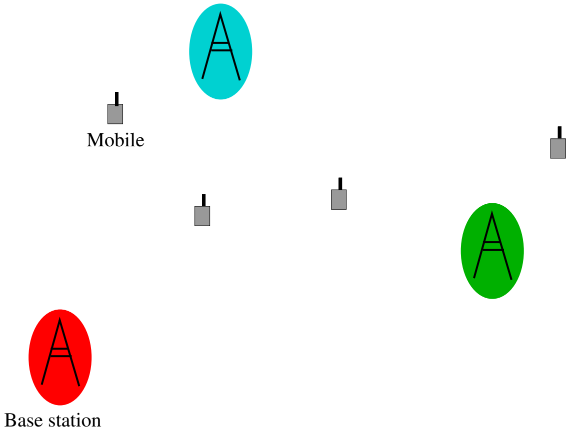

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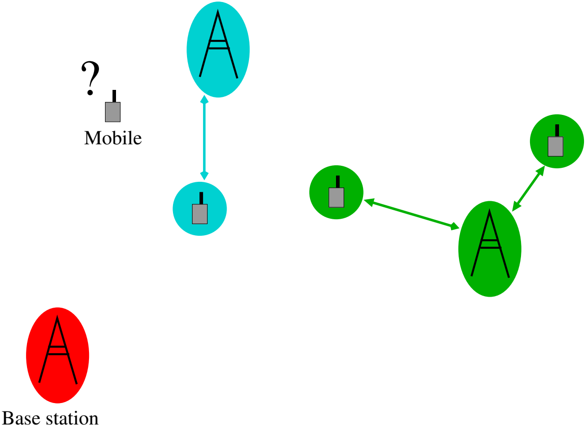

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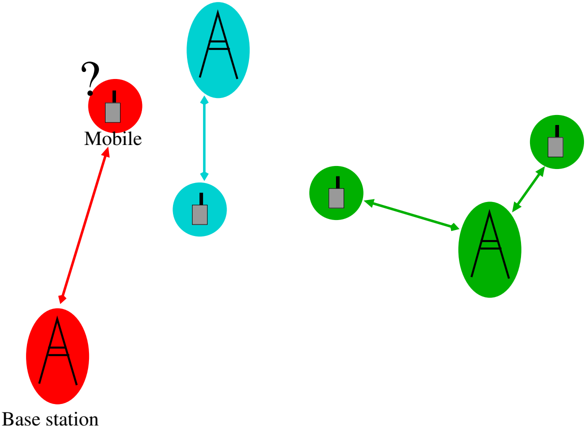

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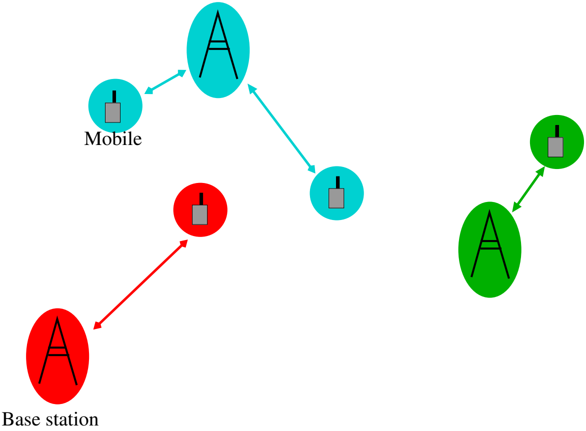

Application: The Optimal Association Problem

Context

- The cells of different technologies overlap (LTE, WiFi, Wimax, etc)
- Mobiles are multi-technology compatible
- Protocols:
  - Multi-homing: having several connections active at once
  - Vertical Handover: to switch from one technology to another

Goal

Find an association algorithm between mobiles and base stations that is:

- distributed
- optimal

Game Formulation

- A set $\mathcal{N}$ of mobiles.
- A set $\mathcal{I}_n$ of Base Stations (BS) that $n \in \mathcal{N}$ can connect to.
- $s_n \in \mathcal{I}_n$ choice of mobile n.
- ℓ^i: load (vector) of BS i: $\ell^i_n = \begin{cases} 1 \text{ if } s_n = 1 \\ 0 \text{ else.} \end{cases}$
- $u_n(\ell^i)$: throughput of mobile n given the load of cell i.

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Association game.

$$(\mathcal{N}, \mathcal{I}, \mathcal{U})$$

Step 1: Creating Fake Games

Original Game.

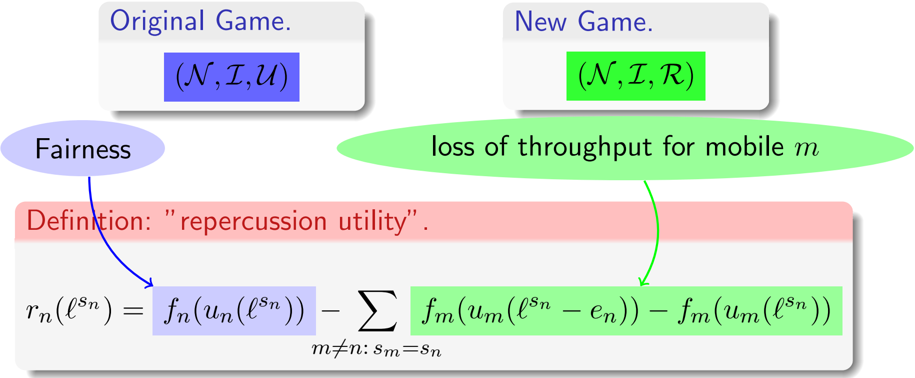
$(\mathcal{N}, \mathcal{I}, \mathcal{U})$

New Game.

$(\mathcal{N}, \mathcal{I}, \mathcal{R})$

How to design games so as to serve one's purpose is the object of mechanism design

Step 1: Creating Fake Games



Original Game.

$(\mathcal{N},\mathcal{I},\mathcal{U})$

New Game.

$(\mathcal{N},\mathcal{I},\mathcal{R})$

Fairness

loss of throughput for mobile m

Definition: "repercussion utility".

$$r_n(\ell^{s_n}) = \boxed{f_n(u_n(\ell^{s_n}))} - \sum_{m\neq n:\, s_m=s_n} \boxed{f_m(u_m(\ell^{s_n}-e_n)) - f_m(u_m(\ell^{s_n}))}$$

Simple computation done by the BS.

How to design games so as to serve one's purpose is the object of mechanism design

Step 2: Choosing a dynamics converging to the Nash equilibria

1. We switch to mixed strategies:

- $q_{n,i} \overset{\text{def}}{=} \mathbb{P}[s_n = i]$.
- $q_n = (q_{n,i})_{i\in\mathcal{I}_n}$: strategy of mobile n.

2. We choose the replicator dynamics:

$$\frac{dq_{n,i}}{dt} = q_{n,i}\Big(\underbrace{\overline{u}_{n,i}(q)}_{\text{expected utility on } i} - \underbrace{\sum_{j\in\mathcal{I}_n} q_{n,j}\overline{u}_{n,j}(q)}_{\text{average expected utility}}\Big).$$

Theorem: (in potential games):

- The replicator dynamics onverges to a set of Nash equilibria.
- Objective function is increasing along the trajectories (Lyapunov function).

Step 3: Deriving a distributed algorithm

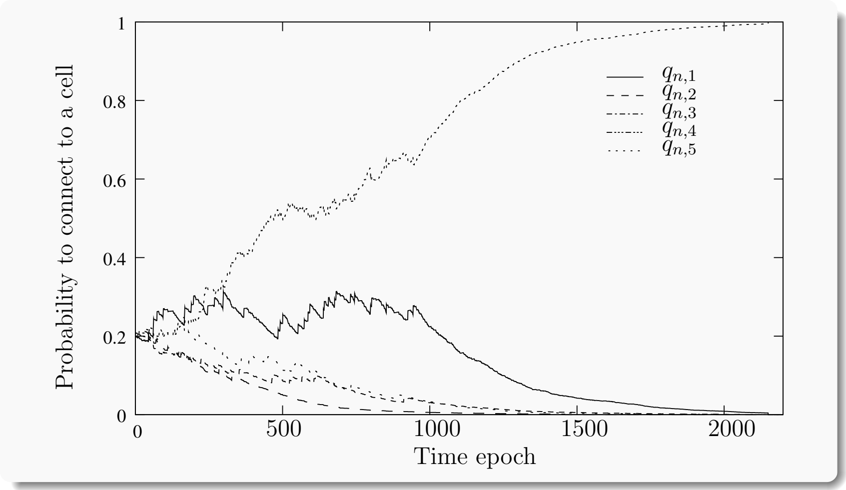
Algorithm: stochastic approximation of the replicator dynamics

For all $n \in \mathcal{N}$:

- Choose initial strategy $q_n(0)$.
- At each time epoch t:
  - Choose s_n according to $q_n(t)$.
  - Update: $q_n(t+1) = q_n(t) + \underbrace{\varepsilon}_{\text{constant step size}} \overbrace{r_n(\ell^{s_n}(s))}^{\text{repercussion utility}} \Big(\underbrace{1_{s_n=i}}_{\begin{cases} 1 \text{ if } s_n = i \\ 0 \text{ otherwise} \end{cases}} - q_{n,i}(t) \Big)$.

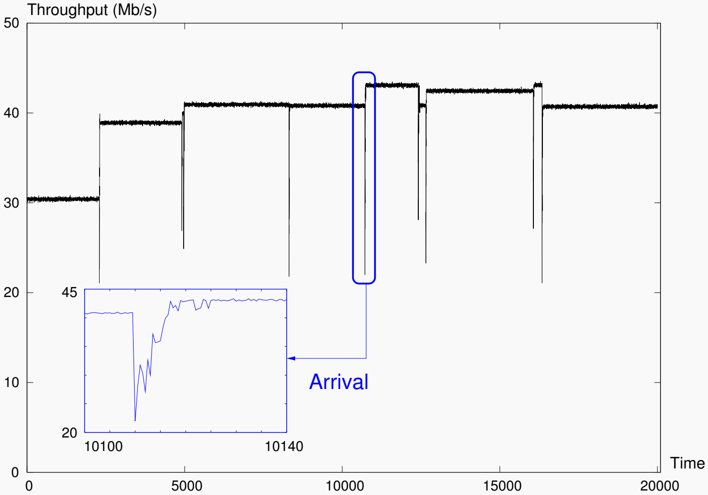
Simple computation for the mobile.

Convergence to Fixed Association for User n



Evolution of one user's strategy that can connect to 5 cells.

Convergence Speed: Dynamic Scenario



Arrivals and departures: evolution of the global throughput with white Gaussian noise.

Outline

1. Individual Versus Collective Interest
    - Matrix Games - Nash Equilibria
    - Population Games - Wardrop Equilibria
    - Conclusion
    - Application: Performance Analysis
2. Designing Efficient Control Mechanisms
    - Objective: Fair Sharing of Resources
    - Direct Method: Protocol Implementation
    - Indirect Method: Modifying the game
3. Conclusion

Résumé

Jeux : situations de décisions interactives dans lesquelles l'utilité (bien-être) de chaque individu dépend des décisions des autres.

Théorie des jeux : théorie de la décision (rationnelle) d'agents stratégiquement interdépendants

Jeux coopératifs	Jeux Non-cooperatifs
Vision globale	Comportement individuel
consensus efficace et équitable	converge (ou non) vers un équilibre

Mécanismes:

recherche de règles du jeu pour obtenir des comportements satisfaisants.

Nous n'avons fait qu'effleurer la surface...

Il existe bien d'autres modèles de jeux:

- Jeux répétés: on rejoue plusieurs fois le même jeu (ex. le tarot en 100 points). Le but est d'alors maximiser le revenu moyen.
- Jeux dynamiques: les joueurs jouent à tour de rôle. L'ensem- ble des stratégies dépend alors des étapes précédentes du jeu.
- Jeux évolutionnaires: inspiré des approches Darwinistes. Se compose d'un jeu interne (entre les individus) et d'un jeu externe (le processus évolutionnaire).
- Jeux stochastiques: jeu dynamique (=évoluant dans le temps) dans lequel les transitions sont probabilistes: le nouvel état est déterminé par une distribution de probabilité dépendant de l'état courant et des actions choisies (Markov Decision Process).
- Équilibres de Stackelberg: jeu entre deux joueurs aux rôles asymétriques: un meneur et un suiveur (utilisé par exemple dans les mécanismes de tarification des e-services). Autres modèles liés: compétition de Bertrand, compétition de Cournot.

Other hot topics in game theory

- Mechanism design: how to design rules of a game so as to achieve a specific outcome, even though each player is selfish.
- Auctions: resource allocation in P2P, frequency allocation in wireless.
- Impact of non-cooperative players in a cooperative environment: free-riders of P2P, UDP clients in TCP networks.
- Fair division or cake cutting problem: how to divide resource such that all recipients believe that they have received their fair share
- Election: Plurality voting systems are not necessarily fair.
- Stable marriages: Problem of finding a matching.
- Super-modular games: utility functions are such that higher choices by one player make one's own strategy higher look relatively more desirable.
- Games with incomplete information or Bayesian games: some player have private information about something relevant to their decision making
- Games with imperfect information: players do not perfectly observe the actions of other players.