Graphs and Networks
The (Δ, D) problem

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May 12, 2011

The (Δ, D) Problem

Network $\longrightarrow$ Graph
Nodes (users, routers, mobiles, towns, web pages) $\longrightarrow$ Vertices
Links (fibers, frequencies, roads, pointers) $\longrightarrow$ Arcs

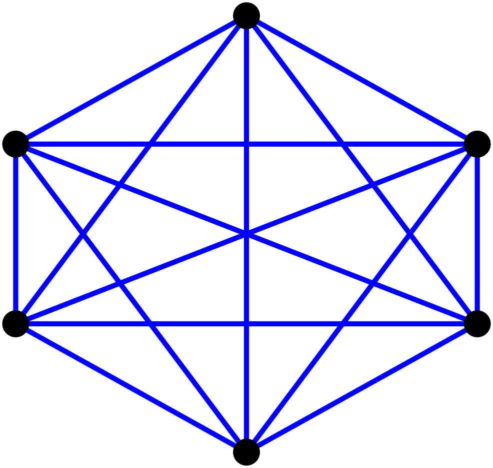
Design a graph

- with degree at most Δ
- with distance between vertices at most D

Objective = maximize the number of vertices $N(\Delta, D)$

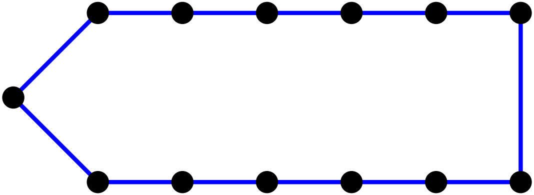
$(\Delta, D = 1)$

Diameter $D = 1 \rightarrow$ Complete Graph $K_{\Delta+1}$



$(\Delta = 2, D)$

Degree $\Delta = 2 \rightarrow$ Cycle C_{2D+1}



A problem of *Le Monde*

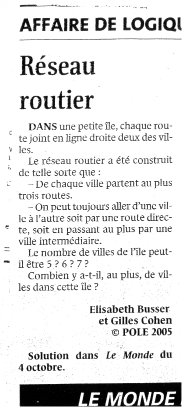
Road Network

Dans une petite île, chaque route joint en ligne droite deux des villes. Le réseau routier a été construit de telle sorte que :

- De chaque ville partent au plus trois routes.
- On peut toujours aller d'une ville à l'autre soit par une route directement, soit en passant au plus par une ville intermédiaire.

Le nombre de villes de l'île peut-il être 5 ? 6 ? 7 ? Combien y a-t-il, au plus, de villes dans cette île ?

Elisabeth Busser et Gilles Cohen Solution dans *Le Monde* du 4 octobre.


AFFAIRE DE LOGIQU

Réseau routier

DANS une petite île, chaque route joint en ligne droite deux des villes.

Le réseau routier a été construit de telle sorte que :

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Elisabeth Busser
et Gilles Cohen
© POLE 2005

Solution dans *Le Monde* du 4 octobre.

LE MONDE

A problem of *Le Monde*

Road Network

UE N° 449

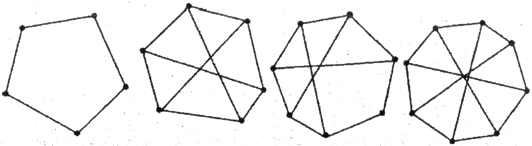
Solution du jeu n° 448 paru dans *Le Monde* du 27 septembre.

Il y a au plus 8 villes dans l'île.

Voici des solutions à 5 villes, à 6 villes, à 7 villes, à 8 villes.

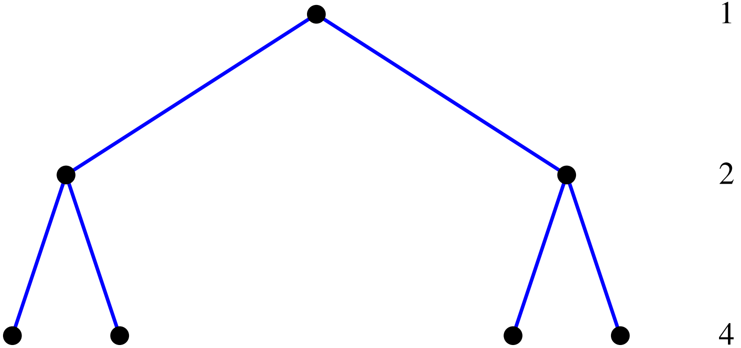
– Avec 9 villes, il ne peut pas se faire que des 9 villes partent 3 routes. En effet, on parviendrait à 27 extrémités de routes (nombre impair), alors que chaque route compte 2 extrémités.

Il existe donc une ville, soit A, dont ne partent que 2 routes. A est reliée directement à B et C, de B et C partent au plus 2 routes menant au plus à D, E, F et G, ce qui fait au total au plus 6 villes reliées directement ou via une étape à A. Il ne peut y en avoir 8.



$(\Delta = 3, D = 2)$

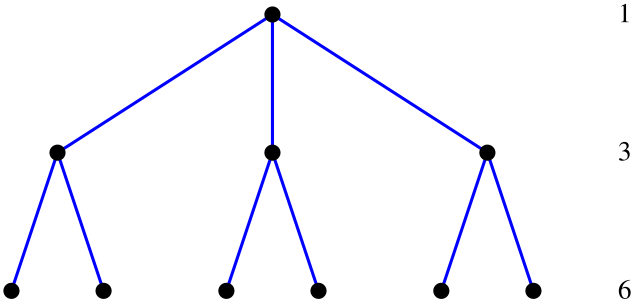
If some vertex has degree 2



$n \leq 7$

$(\Delta = 3, D = 2)$

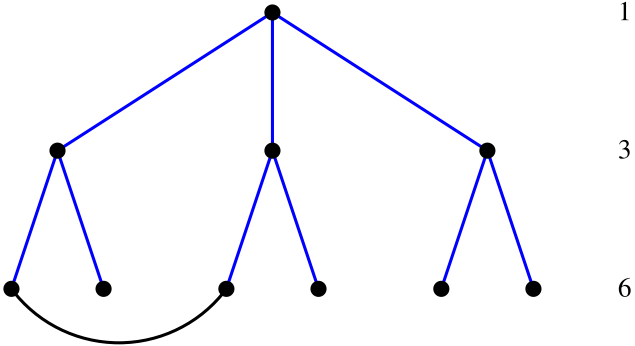
Moore's Graphs



$n \geq 8$ $\quad$ $n \neq 9$ $\quad$ $n \leq 10$

$(\Delta = 3, D = 2)$

Moore's Graphs



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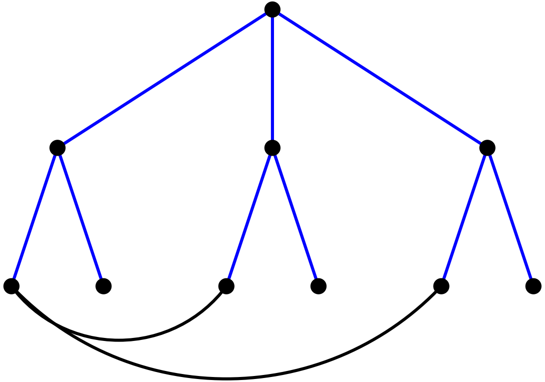


1

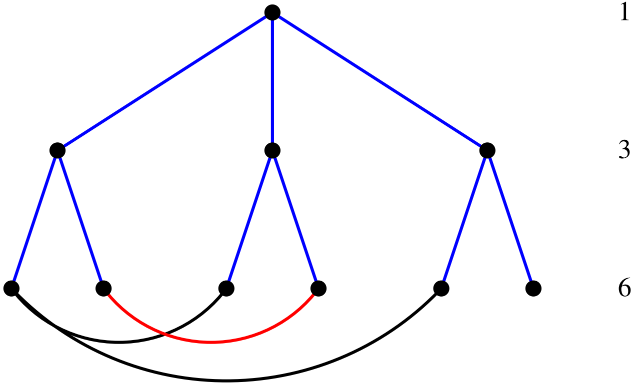
3

6

$n \geq 8$ $\quad$ $n \neq 9$ $\quad$ $n \leq 10$

$(\Delta = 3, D = 2)$

Moore's Graphs



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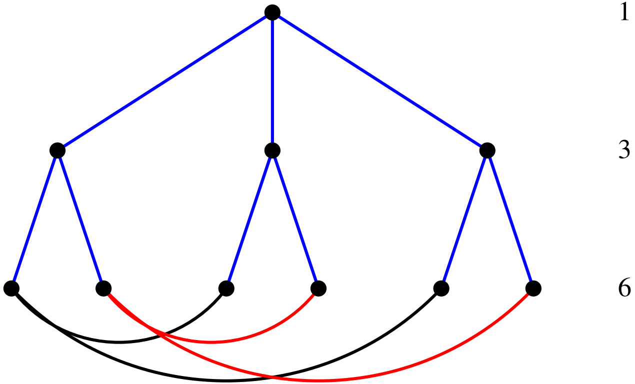

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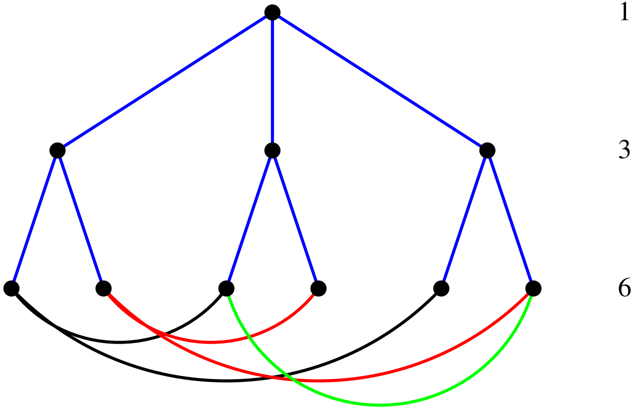

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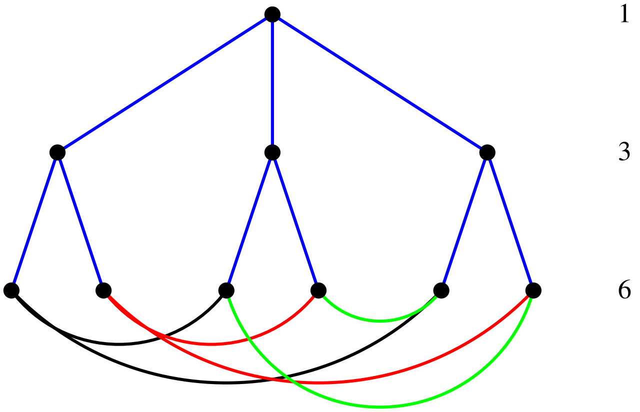
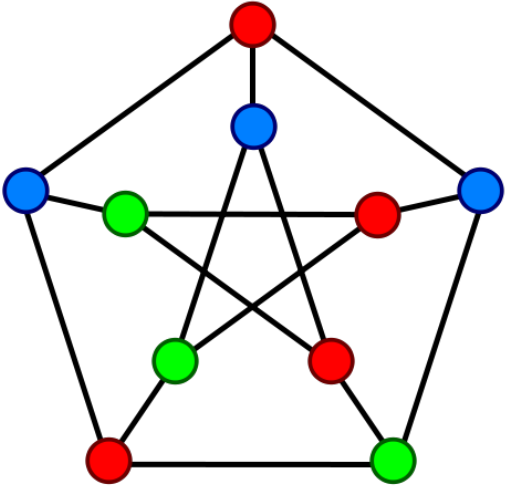


$n \geq 8$ $\quad$ $n \neq 9$ $\quad$ $n = 10$

$(\Delta = 3, D = 2)$

Petersen Graph



Moore's bound

Theorem

$N(\Delta, D) \leq 1 + \Delta + \Delta(\Delta - 1) + \ldots + \Delta(\Delta - 1)^{D-1}$

Bound reached for:

- $(\forall \Delta, D = 1) \Rightarrow K_{\Delta+1}$;
- $(\Delta = 2, \forall D) \Rightarrow C_{2D+1}$;
- $(\Delta = 3, D = 2) \Rightarrow N = 10$ (Petersen);
- $(\Delta = 7, D = 2) \Rightarrow N = 50$ (Hoffman-Singleton);
- $(\Delta = 57, D = 2) \Rightarrow N = 3250$????

Table

http://www.eyal.com.au/wiki/The_Degree/Diameter_Problem

Δ, D	2	3	4	5	6	7
3	**10**	**20**	**38**	70	132	196
4	**15**	41	96	364	740	1320
5	**24**	72	210	624	2772	5516
6	**32**	110	390	1404	7917	19383
7	**50**	168	672	2756	11988	52768
8	57	253	1100	5060	39672	131137
9	74	585	1550	8200	75893	279616
10	91	650	2286	13140	134690	583083
11	104	715	3200	19500	156864	1001268

Table

http://www.eyal.com.au/wiki/The_Degree/Diameter_Problem

Ratios in % to the Moore's bound

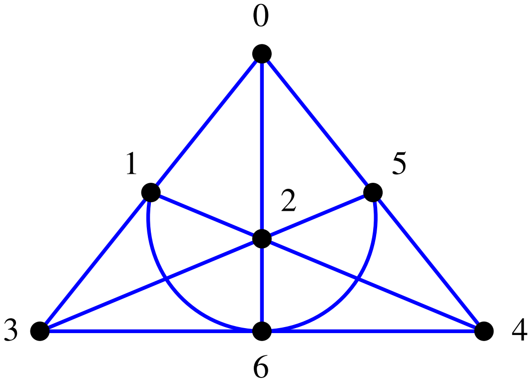
Δ, D	2	3	4	5	6	7
3	**100**	**91**	**83**	74	69	51
4	**88**	77	60	75	51	30
5	**92**	68	49	37	41	20
6	**86**	59	42	30	34	17
7	**100**	56	37	25	18	13
8	88	55	34	23	25	12
9	90	89	29	19	23	10
10	90	71	28	18	20	10
11	85	59	26	16	13	8

$D = 2, 3$

Constructions based on finite geometries

Projective plane of order 2:

013 124 235 346 450 561 602



$\Delta = 3$

$D > log_2(N+2) - log_2(3)$

Random Graph: cycle $C_N + \frac{N}{2}$ chords $\Rightarrow$ good diameter !!

Chung, Bollobás, De la Vega (1982)

Best known $D = 1.47.log_2(N))$

$N = 2^{\frac{D}{1.47}}$ (Moore's bound 2^D)

De Bruijn and Kautz digraphs

Out-degree $\leq d$.

Theorem (Moore's bound)

$N(d, D) \leq 1 + d + \ldots + d^D$

Bound reached only if:

- $d = 1 \Rightarrow$ circuits;
- $D = 1 \Rightarrow$ complete graphs

De Bruijn: $N = d^D$

Kautz: $N = d^D + d^{D-1}$ (optimal for $D = 2$)

De Bruijn digraphs $B(d, D)$

Definition

Vertices: words of length D $(x_1, \ldots, x_D)$ ($x_i \in$ alphabet with d letters);
Arcs: $(x_1, \ldots, x_D) \rightarrow (x_2, \ldots, x_D, *)$

Definition

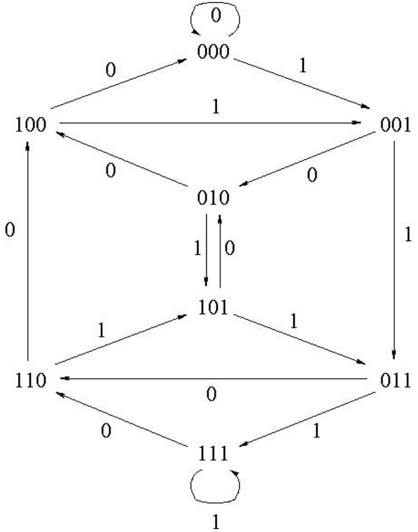
Vertices: integers modulo N;
Arcs: $i \longrightarrow di + \alpha$, $0 \leq \alpha \leq d - 1$
case $d = 2$: $i \rightarrow 2i$ and $i \rightarrow 2i + 1$

Definition

Based on line digraphs

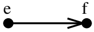
De Bruijn digraphs

Words with 3 letters on an alphabet of 2 letters



Line Digraph

- G $\Rightarrow$ $L(G)$
- arcs $\Rightarrow$ vertices
- 

$\Rightarrow$ 

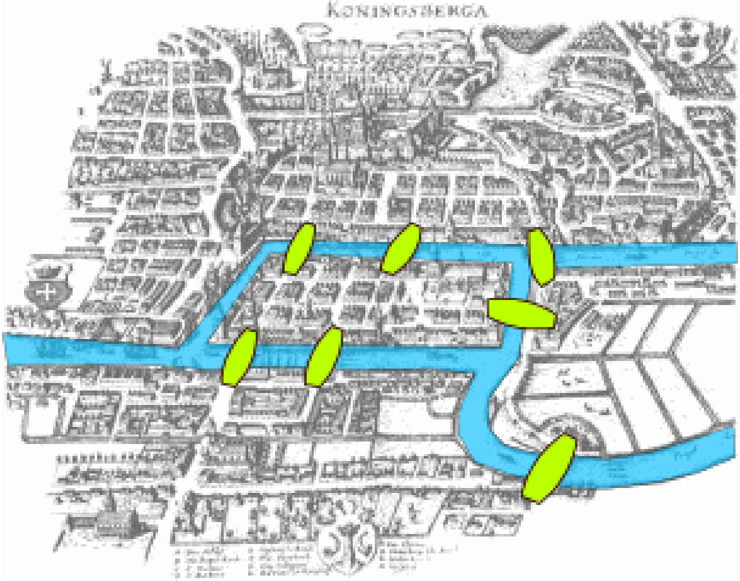
- $\overrightarrow{C_N}$ $\Rightarrow$ $\overrightarrow{C_N}$
- $B(d, D)$ $\Rightarrow$ $B(d, D+1)$
- Diameter D $\Rightarrow$ Diameter $D+1$ (if $d > 1$)
- λ arc connected $\Rightarrow$ λ connected

Eulerian

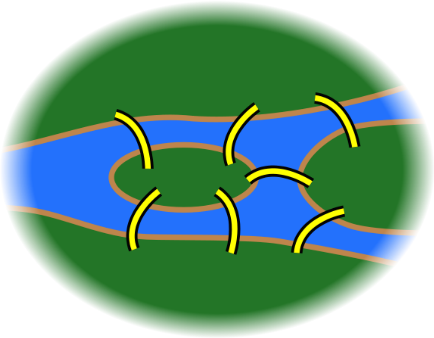
G strongly connected digraph

- Eulerian: circuit containing each arc exactly once.
- Charaterization: $d^+(x) = d^-(x)\ \forall x$
- Polynomial Algorithm
- (Undirected: degrees all even)

Königsberg bridges 1736



Königsberg bridges

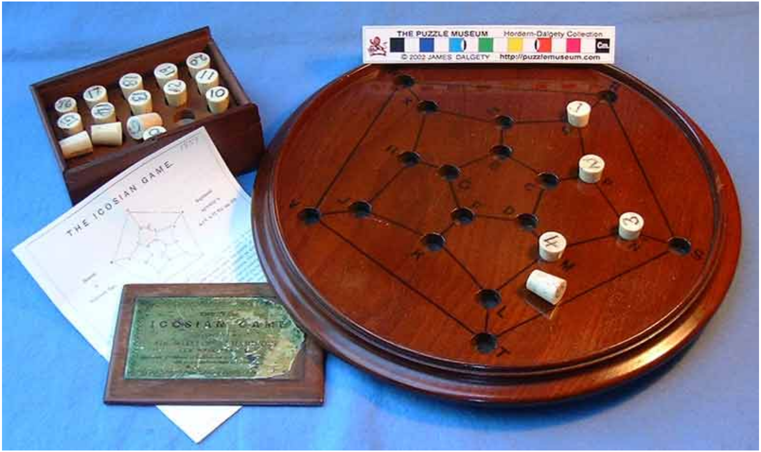


Eulerian / Hamiltonian

G strongly connected digraph

- Eulerian: circuit containing each arc exactly once.
  Charaterization: $d^+(x) = d^-(x)\ \forall x$
  Polynomial Algorithm
- Hamiltonian: circuit containing each vertex exactly once.
  NP-complete Problem

Hamilton Game 1857



Eulerian / Hamiltonian

G strongly connected digraph

- Eulerian: circuit containing each arc exactly once.
  Charaterization: $d^+(x) = d^-(x)\ \forall x$
  Polynomial Algorithm
- Hamiltonian: circuit containing each vertex exactly once.
  NP-complete Problem

G Eulerian digraph $\Leftrightarrow$ $L(G)$ Hamiltonian

$B(d, D)$ Eulerian $\Rightarrow$ $B(d, D+1) = L(B(d, D))$ Hamiltonian

De Bruijn sequences

Circular sequence of length d^D such that every subsequence of length D appears exactly once

Same problem as finding a Hamiltonian circuit in the De Bruijn digraph

0001110100

Cards Trick

????

Cards Trick (1/2)

Idea 1: coding of the cards on 5 bits.

- 1-2 bits for the color:
  00 diamond, 01 heart, 10 club, 11 spade
  0 RED ; 1 BLACK
- 3-5 bits for the value:
  000 for 7, 001 for 8, . . ., 111 for an ace

Examples:
01101 for the queen of heart
11011 for the 10 of spade

Cards Trick (2/2)

Idea 2: hamiltonian cycle of $B(2,5)$
Each subsequence $x_1 x_2 x_3 x_4 x_5 \longrightarrow$ vertex of $B(2,5)$

- $x_1 \ldots x_5 \longrightarrow$ card 1
- $x_2 \ldots x_6 \longrightarrow$ card 2
- $x_3 \ldots x_7 \longrightarrow$ card 3
- $x_4 \ldots x_8 \longrightarrow$ card 4
- $x_5 \ldots x_9 \longrightarrow$ card 5

Card's color $i \Rightarrow x_i$
5 colors (red or black) $\Rightarrow$ then we know the first card
Here the hamiltonian cycle was constructed starting from 00001
and using the rule $x_{i+5} = x_i + x_{i+3}$

Hypergraphs

Bus networks $\rightarrow$ Hypergraph
Hyperedge = subset of vertices
r-uniform: hyperedge of size r
$r = 2 \rightarrow$ Graph

Design an hypergraph

- r-uniform
- of degree at most Δ
- of distance between vertices at most D

Objective = maximize the number of vertices $N(\Delta, D, r)$

Moore's bound

Theorem

$N(\Delta, D, r) \leq$
$1 + \Delta(r-1) + \Delta(\Delta-1)(r-1)^2 + \ldots + \Delta(\Delta-1)^{\Delta-1}(r-1)^{\Delta}$

Tight bound for $D > 2$ (except cycles C_{2D+1})
For $D = 2$, bound reached for $r = 2, \delta = 3, 7, 57$??
$r = 4, \delta = 7$?? (400 vertices, 700 edges)
and some cases for $r > 4$

Theorem

$N(\Delta, 1, r) = 1 + \Delta(r-1) \Rightarrow$ *there exists an $(n, r, 1)$-design*
$=$ *set of n elements, blocks of size r, such that*
each pair of vertices appears in exactly one block
$=$ *partition of the edges of K_n into K_r*

$D = 2, 3$

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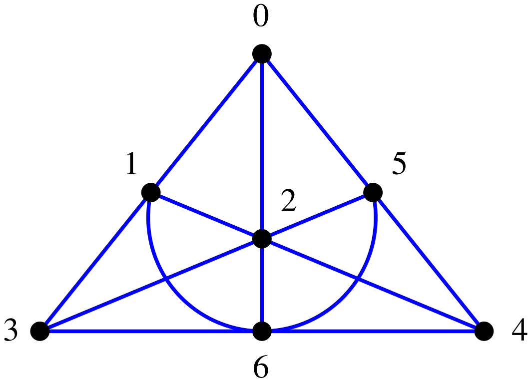
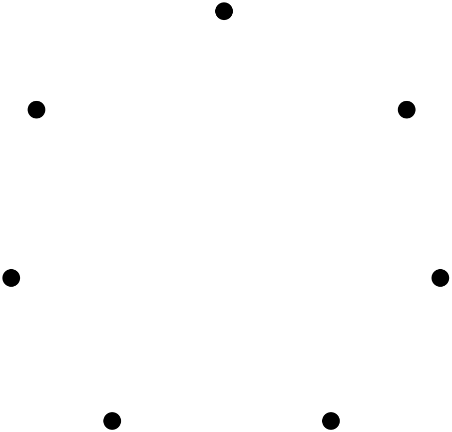


Decomposition of K_7 into triangles



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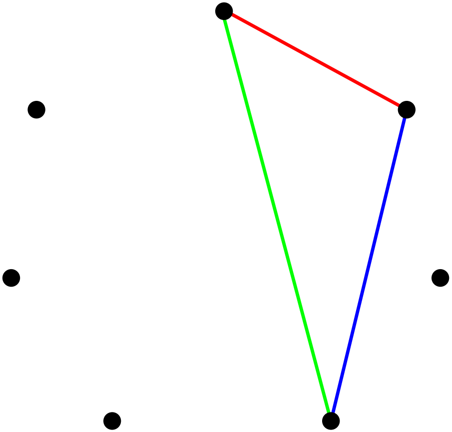

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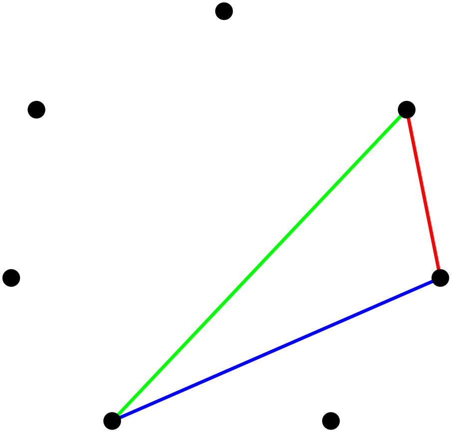

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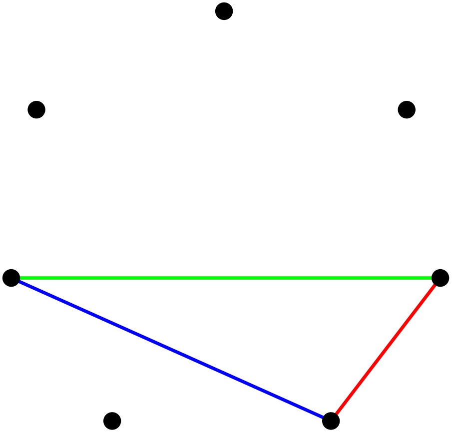

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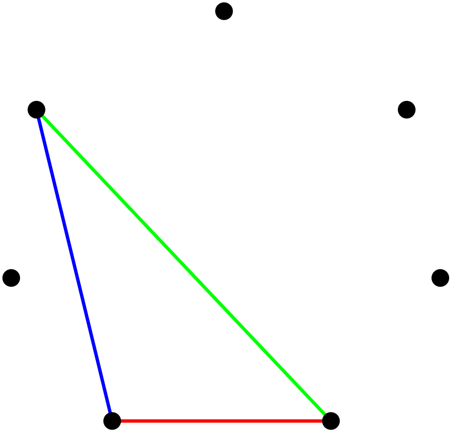

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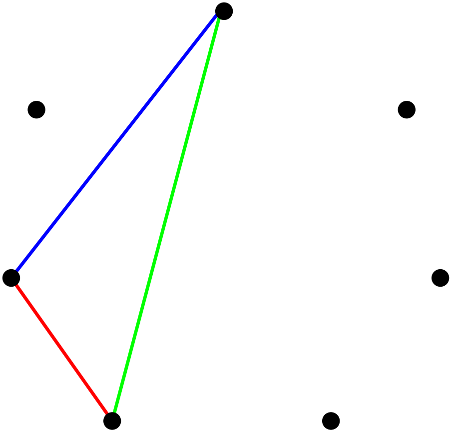

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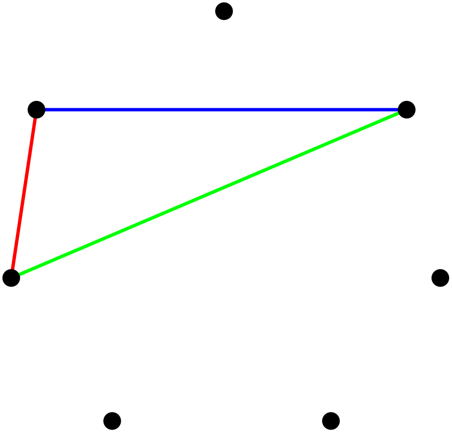

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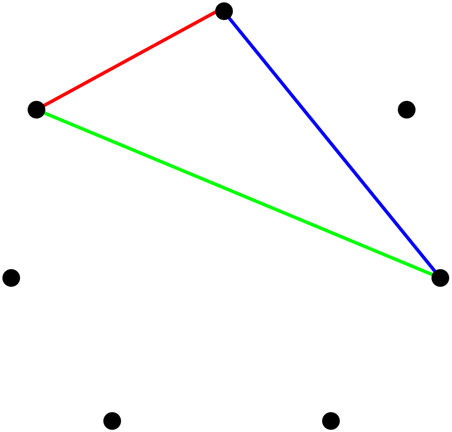

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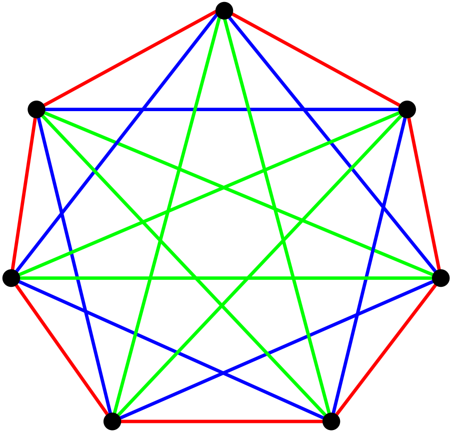
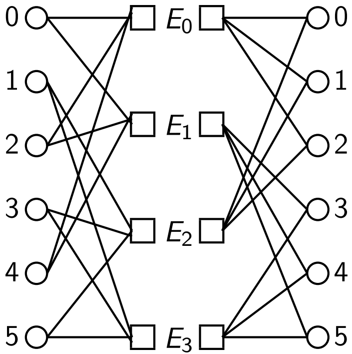


Directed hypergraphs

Hyperarc $= (E^-, E^+)$ E^- inset ; E^+ outset

De Bruijn dihypergraphs



Hypergraphs

Generalization to hypergraphs

Theoretical researchs (1986-1996 + 2011 !)
Connectivity, Line dihypergraphs, ...

Hypergraphs

Generalization to hypergraphs

Theoretical researchs (1986-1996 + 2011 !)
Connectivity, Line dihypergraphs, ...

$\Rightarrow$

Used in the super computer EKA (TATA Industries) :
4th in 2008 (1st in Asia)

http://en.wikipedia.org/wiki/EKA_(supercomputer)

Conclusion

- Eka uses an interconnect designed using concepts from projective geometry. (l'auteur renvoie à notre article)
- The details of the interconnect are beyond the scope of this article. (Translation: I did not understand the really complex mathematics that goes on in those papers. Suffice it to say that before they are done, fairly obscure branches of mathematics get involved).
- This interconnect gives linear speedup for applications but the complexity of building the interconnect increases only near-linearly. The upshot of this is that to achieve a given application speed (i.e. number of Teraflops), Eka ends up using fewer nodes than its compatriots

MERCI