Course Material

Week 3: Extended Kalman Filters

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3.1. Examples for the Action in the EKF

$$\begin{cases}\mu_a = f(\mu, u^m)\\ P_a = F_x P F_x^T + F_u Q F_u^T\end{cases}$$

$x = N(\mu, \sigma_x^2) \qquad P = \sigma_x^2$

$\alpha = N(\alpha^m, \sigma_\alpha^2) \qquad Q = \sigma_\alpha^2$

$S = x$

$x_{i+1} = x_i + R\,\underbrace{\alpha_{i+1}}_{u_{i+1}}$

$F_x = 1 \qquad F_u = R$

$\mu_a = \mu + R\alpha^m$

$P_a = \sigma_x^2 + R^2\sigma_\alpha^2$

Screen 1: Encoders in 1D

$S = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$

$S_{i+1} = f(S_i, u_{i+1})$

$$\begin{cases} x_{i+1} = x_i + \delta\rho\cos\theta_i \\ y_{i+1} = y_i + \delta\rho\sin\theta_i \\ \theta_{i+1} = \theta_i + \delta\theta \end{cases}$$

$\delta\rho = \dfrac{S_R + S_L}{2}$

$\delta\theta = \dfrac{S_R - S_L}{B}$

$u^m = \begin{bmatrix} S_R^m \\ S_L^m \end{bmatrix} \qquad u = N\left(\begin{bmatrix} S_R^m \\ S_L^m \end{bmatrix}, Q\right) \qquad Q = \begin{bmatrix} K^R|S_R| & \emptyset \\ \emptyset & K^L|S_L| \end{bmatrix}$

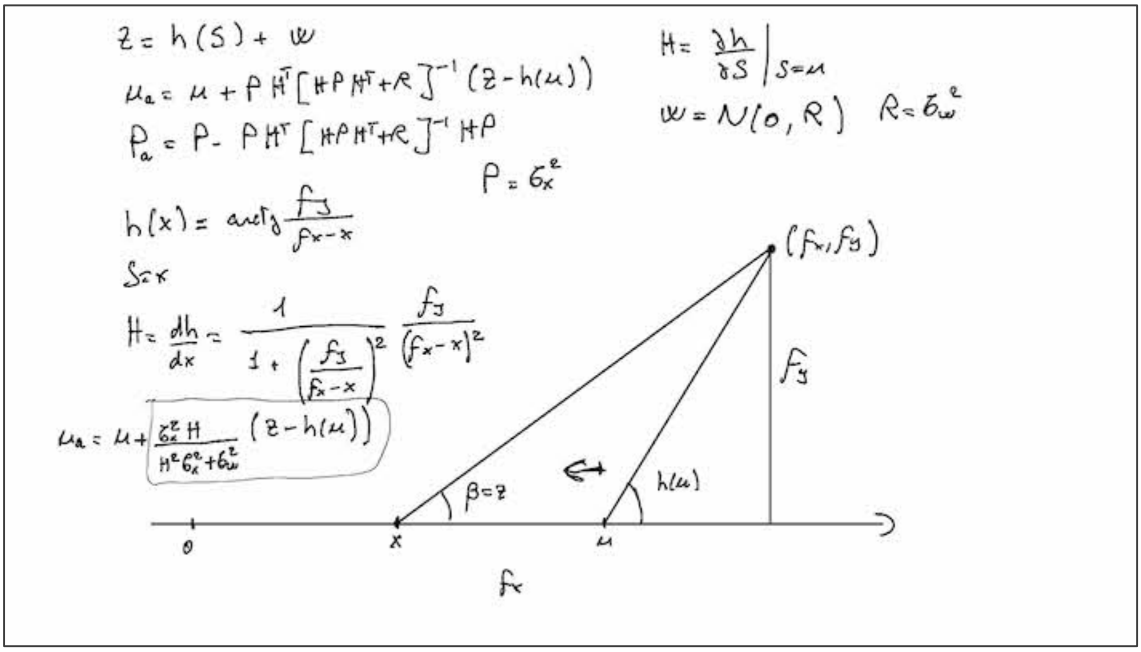
CHONG-KLEEMAN

$$\begin{cases}\mu_a = f(\mu, u^m)\\ P_a = F_x P F_x^T + F_u Q F_u^T\end{cases}$$

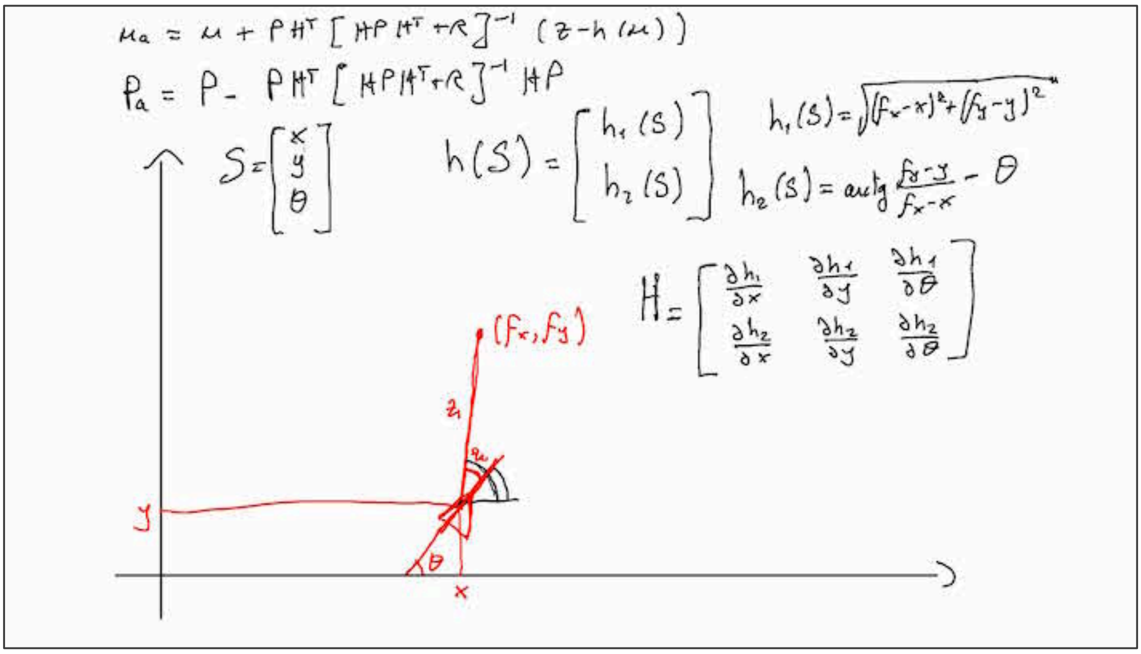
$F_{x_{3\times3}} = \begin{bmatrix} 1 & 0 & -\delta\rho\sin\theta \\ 0 & 1 & \delta\rho\cos\theta \\ 0 & 0 & 1 \end{bmatrix} \qquad F_{u_{3\times2}} = \begin{bmatrix} \frac{\cos\theta}{2} & \frac{\cos\theta}{2} \\ \frac{\sin\theta}{2} & \frac{\sin\theta}{2} \\ \frac{1}{B} & -\frac{1}{B} \end{bmatrix}$

Screen 2: Encoders in 2D

3.2. Examples for the Perception in the EKF



Screen 1: Bearing of a point feature when the robot moves in 1D



Screen 2: Bearing and distance of a point features in 2D

3.3. The EKF is a weight mean

$x \quad z_1 \quad z_2 \qquad z_1 = N(x, \sigma_1^2) \qquad z_2 = N(x, \sigma_2^2)$

$\hat{x} = P_1 z_1 + P_2 z_2 \qquad P_1 + P_2 = 1 \qquad P_1 \propto \frac{1}{\sigma_1^2} \qquad P_2 \propto \frac{1}{\sigma_2^2}$

$$\boxed{\hat{x} = \frac{\frac{1}{\sigma_1^2} z_1 + \frac{1}{\sigma_2^2} z_2}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}} = \frac{\sigma_2^2 z_1 + \sigma_1^2 z_2}{\sigma_1^2 + \sigma_2^2}}$$

$$\hat{\sigma}^2 \qquad \frac{1}{\hat{\sigma}^2} = \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \qquad \boxed{\hat{\sigma}^2 = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}}$$

Screen 1: Simple weight mean

$$\mu_a = \mu + P H^T [H P H^T + R]^{-1} (z - h(\mu))$$

$$P_a = P - P H^T [H P H^T + R]^{-1} H P$$

$\mu \to z_1$

$z \to z_2$

$h =$ identity function $\qquad H = 1$

$P \to \sigma_1^2 \qquad \hat{x} = z_1 + \sigma_1^2 \cdot 1 [\sigma_1^2 + \sigma_2^2]^{-1} (z_2 - z_1) =$

$R \to \sigma_2^2 \qquad z_1 + \frac{\sigma_1^2 z_2 - \sigma_1^2 z_1}{\sigma_1^2 + \sigma_2^2} = \frac{z_1 \sigma_2^2 + z_2 \sigma_1^2}{\sigma_1^2 + \sigma_2^2}$

$\mu_a \to \hat{x}$

$P_a \to \hat{\sigma}^2 \qquad \hat{\sigma}^2 = \sigma_1^2 - \sigma_1^2 [\sigma_1^2 + \sigma_2^2]^{-1} \sigma_1^2 = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}$

Screen 2: EKF perception for a special case: 1D and h is the identity function

3.4. The use of the EKF in robotics

$$\begin{cases} \mu_a = f(\mu, u^m) \\ P_a = F_x P F_x^T + F_u Q F_u^T \end{cases}$$

ACTION

$$\begin{cases} \mu_a = \mu + P H^T [H P H^T + R]^{-1} (z - h(\mu)) \\ P_a = P - P H^T [H P H^T + R]^{-1} H P \end{cases}$$

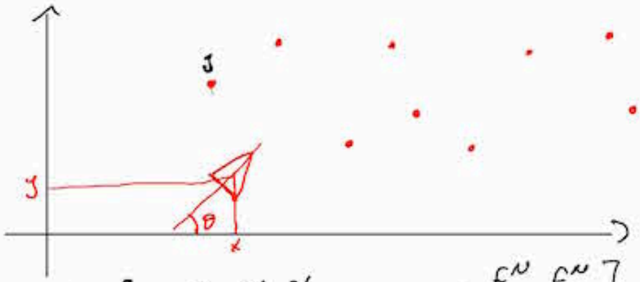
PERCEPTION

1) Introduce a suitable frame and the state to be estimated
2) Derive the analytical expression of the functions "f" and "h"
3) Compute the Jacobians $F_x = \frac{\partial f}{\partial S}$ $F_u = \frac{\partial f}{\partial u}$ $H = \frac{\partial h}{\partial S}$

- LOCALIZATION
- SIMULTANEOUS LOCALIZATION AND MAPPING (SLAM)
- COOPERATIVE LOCALIZATION
- SELF-CALIBRATION

Screen 1: Basic steps to implement an EKF and important estimation problems in robotics

3.5 Simultaneous Localization and Mapping (SLAM)



$F_{x\,(3+2N)\times(3+2N)}$

$F_{u\,(3+2N)\times 2}$

$h(S)=h(x\,y\,\theta\,f_x^j\,f_y^j)$

$H=\left[\frac{\partial h}{\partial x}\,\frac{\partial h}{\partial y}\,\frac{\partial h}{\partial \theta},\,\emptyset\emptyset,\,\frac{\partial h}{\partial f_x^j}\,\frac{\partial h}{\partial f_y^j},\,\emptyset\emptyset\right]$

$3+2N$

$S=\left[x\,y\,\theta\;\; f_x^1\, f_y^1\,, -----, f_x^N, f_y^N\right]$

$f \quad h$

$$\begin{cases} x_{i+1} = x_i + \delta\rho\cos\theta_i \\ y_{i+1} = y_i + \delta\rho\sin\theta_i \\ \theta_{i+1} = \theta_i + \delta\theta \\ f^1_{x\,i+1} = f^1_{x\,i} \\ \vdots \\ f^N_{y\,i+1} = f^N_{y\,i} \end{cases}$$

$$F_x=\begin{bmatrix} F^R_{x\,3\times3} & 0_{3\times 2N} \\ 0_{2N\times3} & 1_{2N\times2N} \end{bmatrix}$$

$$F_u=\begin{bmatrix} F^R_{u\,3\times2} \\ 0_{2N\times2} \end{bmatrix}$$

Screen 2: Basic equations for the 2D case with N point features: the function f and h and their Jacobians

$$S=\left[S^R, f^1, f^2, ---, f^N\right]$$

$$F_x=\begin{bmatrix} F_x^R & 0 \\ 0 & 1 \end{bmatrix} \qquad F_u=\begin{bmatrix} F_u^R \\ 0 \end{bmatrix} \qquad P=\begin{bmatrix} P_R & P_{RN} \\ P_{RN}^T & P_N \end{bmatrix}$$

$$H=\left[\frac{\partial h}{\partial S^R}, 0, 0, ---\frac{\partial h}{\partial f^j}, 0, 0\right]$$

$$\begin{cases} \mu_a = f(\mu, u) & \text{cost independent of } N \\ P_a = F_x P F_x^T + F_u Q F_u^T = \begin{bmatrix} F_x^R P_R F_x^{RT} & F_x P_{RN} \\ P_{RN}^T F_x^T & P_N \end{bmatrix} + \begin{bmatrix} F_u^R Q F_u^{RT} & 0 \\ 0 & 0 \end{bmatrix} \end{cases}$$

↳ cost depend linearly on N

$O(N)$

Screen 1: Matrices to implement the EKF and computational cost of the action

$$\mu_a = \mu + PH^T[HPH^T+R]^{-1}(z-h(\mu)) \quad \text{cost is } O(N)$$

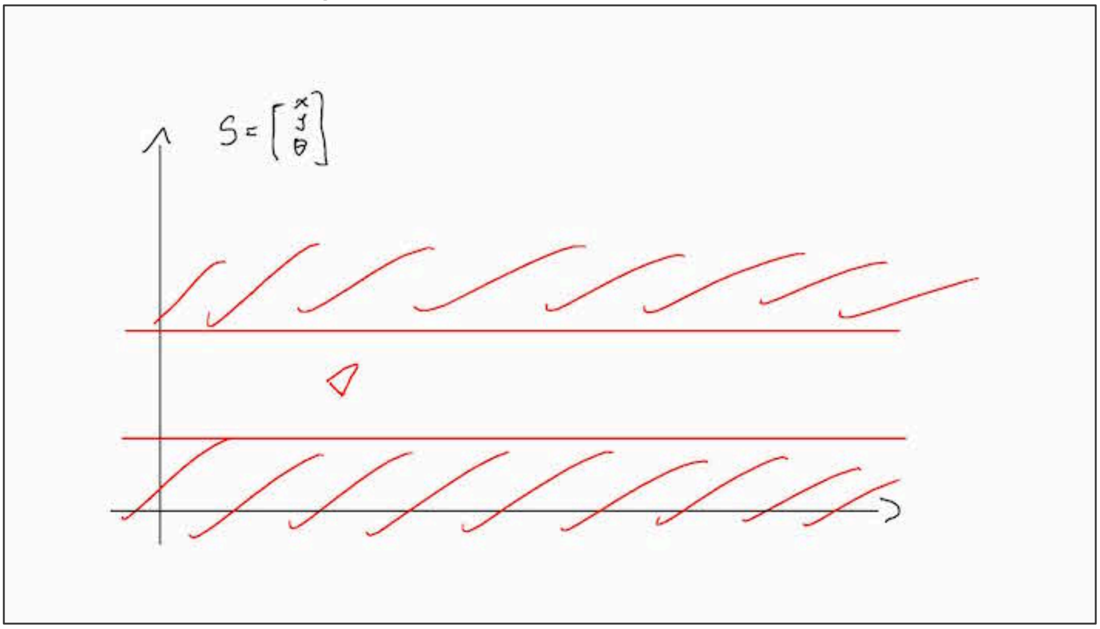
$$P_a = P - PH^T[HPH^T+R]^{-1}HP \quad \text{cost is } O(N^2)$$

$$H = [H^R, \emptyset\ \emptyset, ---, H^J, \emptyset\ \emptyset]$$

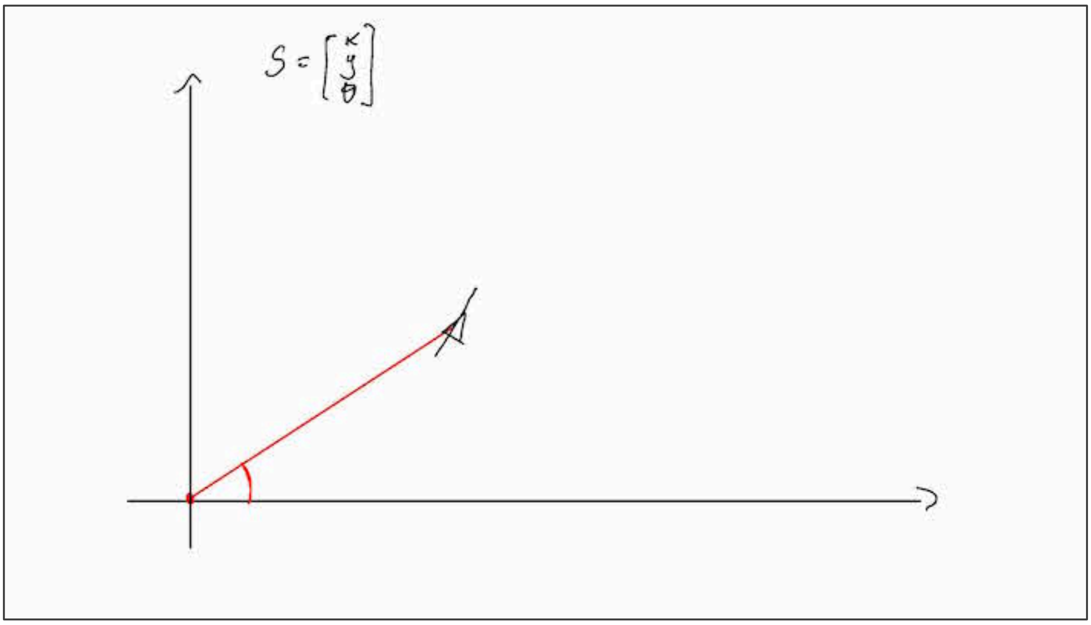
$$\begin{bmatrix} P_R & P_{Rm} \\ P_{Rm}^T & P_m \end{bmatrix} \begin{bmatrix} H^{R\,T} \\ 0 \\ 0 \\ H^{J\,T} \\ 0 \end{bmatrix} \quad O(N)$$

Screen 2: Computational cost of the perception

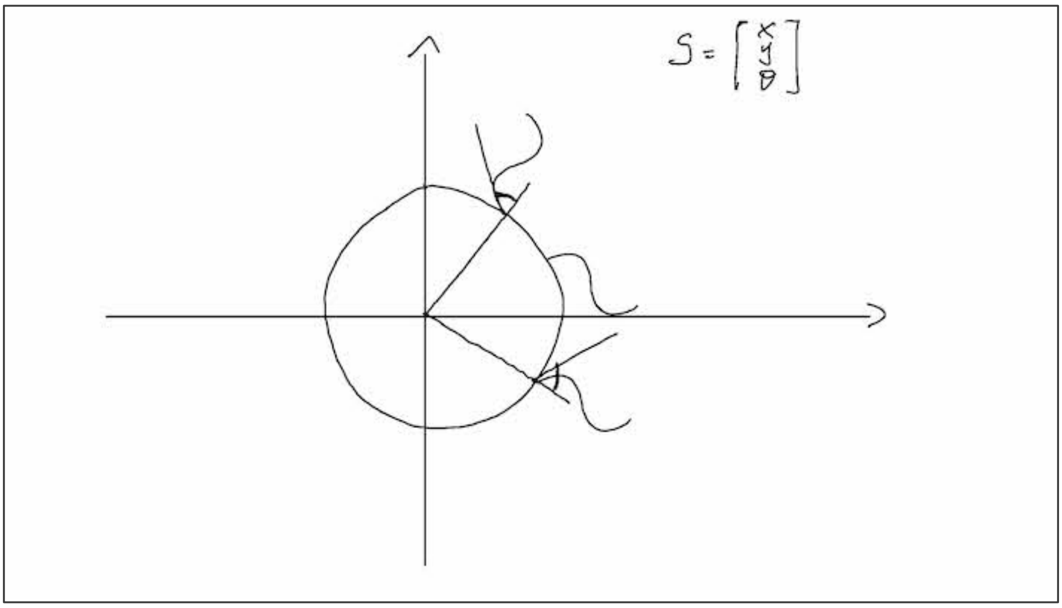
3.6 Observability in robotics



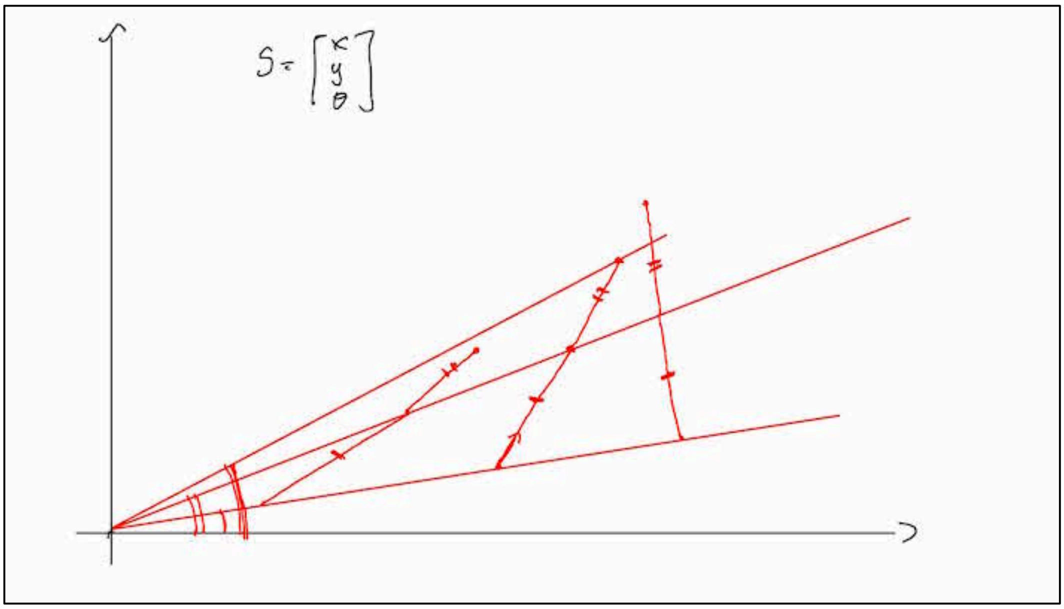
Screen 1: Uniform hallway: intuitively, we cannot estimate the x-coordinate



Screen 2: 2D case with range sensor that provides the distance from the origin



Screen 3: The 2 trajectories (the upper and the lower) provide the same measurements



Screen 4: All the three trajectories provide a different third bearing measurement

3.7. Observability Rank Criterion

OBSERVABILITY RANK CRITERION

HERMAN KRENER 1977

$$\begin{cases} x_{i+1} = x_i + \delta\rho \cos\theta_i \\ y_{i+1} = y_i + \delta\rho \sin\theta_i \\ \theta_{i+1} = \theta_i + \delta\theta \end{cases} \qquad \begin{aligned} \delta\rho &= v\, dt \\ \delta\theta &= \omega\, dt \end{aligned}$$

$$\dot{S} = f_0(S) + \sum_{i=1}^{m} u_i f_i(S)$$

$$\begin{cases} \dot{x} = v\cos\theta \\ \dot{y} = v\sin\theta \\ \dot{\theta} = \omega \end{cases} \qquad f_0(S) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad m = 2 \quad \begin{aligned} u_1 &= v \\ u_2 &= \omega \end{aligned}$$

$$f_1(S) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \qquad f_2(S) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Screen 1: Continuous time model for a wheeled robot in 2D (the unicycle)

$$\dot{S} = f_0(S) + \sum_{i=1}^{m} u_i f_i(S) \qquad h(S)$$

$$\mathcal{L}^0 h = h$$

$m+1$ first order $\quad \mathcal{L}^1_0 h = (\nabla h)\cdot f_0 \; --- \; \mathcal{L}^1_i h = (\nabla h)\cdot f_i$

$(m+1)^2$ second order $\quad \mathcal{L}^2_{ij} h = (\nabla \mathcal{L}^1_i h)\cdot f$

$$\Omega^0 = \text{span}\{\nabla h\}$$

Ω^1 = span of the gradients of all the Lie derivatives up to order 1

Screen 2: Lie derivatives and the observable codistribution

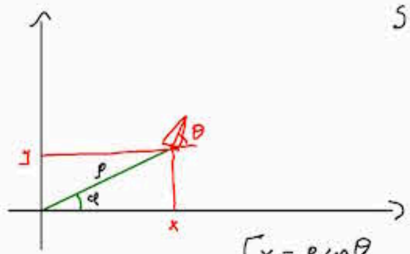
$$\Omega^0 \subset \Omega^1 \subset \Omega^2 \; ---$$

n = dimension of $S \qquad \dim(\Omega^k) \leq n$

$$\dim(\Omega^j) = \dim(\Omega^{j+1}) \leq n$$

Screen 3: Dimension of the observable codistribution

3.8. Applications of the Observability Rank Criterion



$$S = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \qquad \begin{cases} \dot{x} = v \cos\theta \\ \dot{y} = v \sin\theta \\ \dot{\theta} = \omega \end{cases}$$

1) $h(S) = \sqrt{x^2+y^2} = \rho$

2) $h(S) = \operatorname{atan} \frac{y}{x} = \varphi$

$$S = \begin{bmatrix} \rho \\ \varphi \\ \theta \end{bmatrix} \qquad \begin{cases} x = \rho\cos\theta \\ y = \rho \sin\theta \end{cases} \rightarrow \begin{cases} \rho = \sqrt{x^2+y^2} \\ \theta = \operatorname{atan} \frac{y}{x} \end{cases}$$

$$\begin{cases} \dot{\rho} = v\cos(\theta-\varphi) \\ \dot{\varphi} = \frac{v}{\rho}\sin(\theta-\varphi) \\ \dot{\theta} = \omega \end{cases}$$

$$\dot{S} = f_0(S) + \sum_{i=1}^{m} u_i f_i(S) \qquad u_1 = v \quad u_2 = \omega$$

$$f_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad m = 2 \qquad f_1 = \begin{bmatrix} \cos(\theta-\varphi) \\ \sin(\theta-\varphi) \\ 0 \end{bmatrix} \qquad f_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Screen 1: 2D robot (satisfying the unicycle dynamics) in polar coordinates

1) $f_1 = \begin{bmatrix} \cos(\theta-\varphi) \\ \frac{\sin(\theta-\varphi)}{\rho} \\ 0 \end{bmatrix} \qquad f_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad h(S) = \rho$

$$S = \begin{bmatrix} \rho \\ \varphi \\ \theta \end{bmatrix}$$

$\mathcal{L}^0 h = \rho \qquad [1 \quad 0 \quad 0]$

$\mathcal{L}^1_1 h = \cos(\theta-\varphi) \qquad [0, \ \sin(\theta-\varphi), \ -\sin(\theta-\varphi)]$

$\mathcal{L}^1_2 h = \emptyset$

$\mathcal{L}^2_{11} h = \frac{\sin(\theta-\varphi)^2}{\rho} \qquad [A, \ *, \ -*]$

$\mathcal{L}^2_{12} h = -\sin(\theta-\varphi)$

$\dim(\Omega^1) = 2 < 3$

$\dim(\Omega^2) = \dim(\Omega^1)$

Screen 2: The case when the sensor provides the distance from the origin. The dimension of the observable codistribution is 2<3 and the system is non-observable

2) $f_1 = \begin{bmatrix} \cos(\theta-\varphi) \\ \frac{\sin(\theta-\varphi)}{\rho} \\ 0 \end{bmatrix} \quad f_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad h(S) = \varphi$

$$L^0 h = \varphi \qquad \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

$$L^1_1 h = \frac{\sin(\theta-\varphi)}{\rho} \qquad \begin{bmatrix} -\frac{\sin(\theta-\varphi)}{\rho^2}, & -\frac{\cos(\theta-\varphi)}{\rho}, & \frac{\cos(\theta-\varphi)}{\rho} \end{bmatrix}$$

$$L^1_2 h = \emptyset$$

$$L^2_{12} h = \frac{\cos(\theta-\varphi)}{\rho} \qquad \begin{bmatrix} -\frac{\cos(\theta-\varphi)}{\rho^2}, & \frac{\sin(\theta-\varphi)}{\rho}, & -\frac{\sin(\theta-\varphi)}{\rho} \end{bmatrix}$$

$$\dim(\Omega^2) = 3$$

Screen 3: The case when the sensor provides the bearing of the robot. The dimension of the observable codistribution is 3 and the system is observable