3. Message Attack (ISD)

1. From Generic Decoding to Syndrome Decoding
2. Combinatorial Solutions: Exhaustive Search and Birthday Decoding
3. Information Set Decoding: the Power of Linear Algebra
4. Complexity Analysis
5. Lee and Brickell Algorithm
6. Stern/Dumer Algorithm
7. May, Meurer, and Thomae Algorithm
8. Becker, Joux, May, and Meurer Algorithm
9. Generalized Birthday Algorithm for Decoding
10. **Decoding One Out of Many**

Decoding One Out of Many (DOOM)

N-Syndrome Decoding

Instance: $S \subset \{0,1\}^{n-k}$, $|S| = N$, $H \in \{0,1\}^{(n-k)\times n}$, an integer $w > 0$
Answer: $e \in \{0,1\}^n$ such that $eH^T \in S$ and $\mathrm{wt}(e) \leq w$

We will denote $\mathrm{CSD}_N(H, S, w)$ the set of all solutions to the above problem

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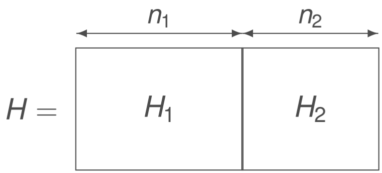
Improvement:

- we get the N solutions at the expense of a factor $\approx \sqrt{N}$
- or we get one solution with a gain of a factor $\approx \sqrt{N}$

Birthday Decoding With Multiple Instances

Solve $\mathrm{CSD}_N(H, S, w)$ with birthday decoding

$$\text{Let} \begin{cases} \mathcal{L}_1 = \{e_1 H_1^T \mid \mathrm{wt}(e_1) = w_1\} \\ \mathcal{L}_2 = \{s + e_2 H_2^T \mid s \in S, \mathrm{wt}(e_2) = w_2\} \end{cases}$$



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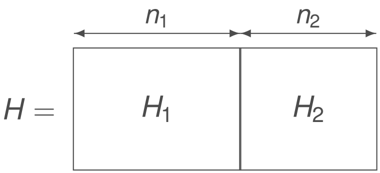

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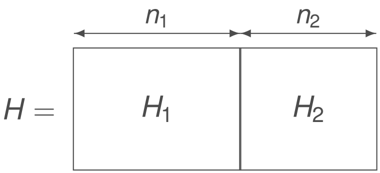

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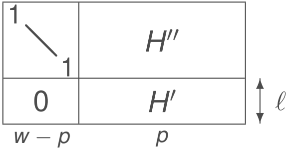
Claim: If $N \leq \binom{n}{w}$, we obtain all solutions of $\mathrm{CSD}_N(H, S, w)$

for a cost $\sqrt{N\binom{n}{w}} + \frac{N\binom{n}{w}}{2^{n-k}}$ (up to a polynomial factor)

DOOM-ISD

Solve $\mathrm{CSD}_N(H, S, w)$ when $S \subset \{eH^T \mid \mathrm{wt}(e) = w\}$ with Dumer Algorithm

The problem has N solutions and we only want one



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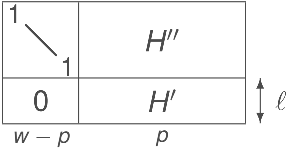

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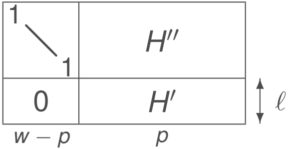

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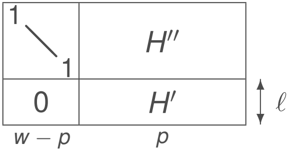

A specific solution requires $\mathcal{N}_\infty = \dfrac{\binom{n}{w}}{\binom{n-k-\ell}{w-p}\binom{k+\ell}{p}}$ iterations

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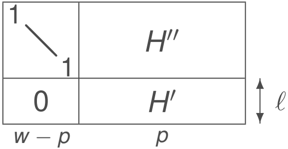

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$$\rightarrow \text{WF}_{\text{DOOM}} = \min_{0 \leq p \leq w} \frac{\binom{n}{w}}{\binom{n-k-\ell}{w-p}\sqrt{N\binom{k+\ell}{p}}} \text{ with } \ell = \log_2 \sqrt{N\binom{k+\ell}{p}}$$

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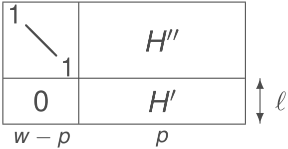

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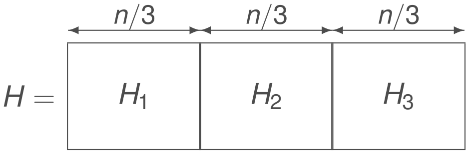
$\rightarrow$ gain of a factor $\approx \sqrt{N}$ as long as $N \leq \min\left(\mathcal{N}_\infty, \binom{k+\ell}{p}\right)$

DOOM-GBA

A gain is also possible with an Order 2 GBA Decoding when $N = |S| = \binom{n/3}{w/3}$



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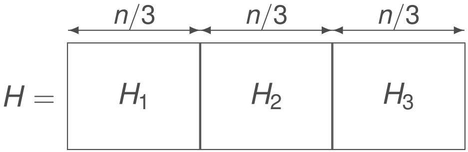

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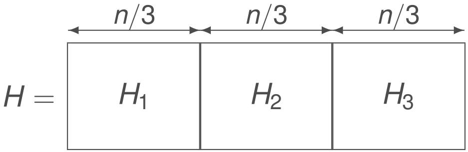

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$$e_1 H_1^T + e_2 H_2^T + e_3 H_3^T + s = 0, s \in S$$

and we have $e = (e_1, e_2, e_3) \in \mathrm{CSD}_N(H, S, w)$

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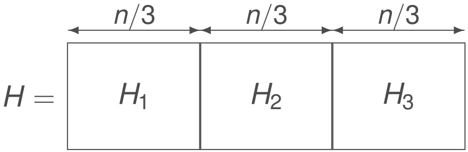

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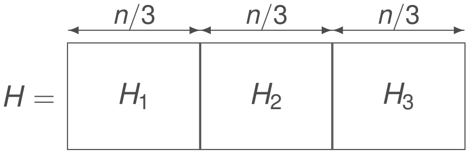

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To be compared with $\sqrt{\binom{n}{w}}$ with the birthday decoding, gaining a factor $\approx \sqrt{N}$

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Code-Based Cryptography